# Estimation of a scale second-order parameter related to the PORT methodology*

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**Abstract**

For heavy right tails and under a semi-parametric framework, we introduce a class of location invariant estimators of a scale second-order parameter and study its asymptotic non-degenerate behaviour. This class is based on the PORT methodology, with PORT standing for peaks over random thresholds. The consistency and asymptotic normality of the new class of estimators is achieved under a third-order condition on the right tail of the underlying model $F$ for intermediate and large ranks, respectively. An illustration of the finite sample behaviour of the estimators is provided through a Monte-Carlo simulation study.

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# 1 Introduction

Our interest lies in heavy right tails, i.e. we are dealing with a random sample $\underline{X}_n = (X_1, \ldots, X_n)$ from an underlying distribution function (d.f.) $F$ with a regularly varying right tail. This means that, for a positive real $\xi$, the right tail-function

$$\overline{F}(x) := 1 - F(x)$$

is such that

$$\lim_{t\to\infty} \overline{F}(tx)/\overline{F}(t) \;=\; x^{-1/\xi}, \quad \text{for all } x > 0, \tag{1.1}$$

i.e. $\overline{F}$ is a regularly varying function at infinity with an index of regular variation equal to $-1/\xi$, $\xi > 0$. We then use the notation $\overline{F} \in RV_{-1/\xi}$.

Let us define

$$G_\xi(x) := \begin{cases} \exp\left(-(1+\xi\, x)^{-1/\xi}\right), & 1+\xi\, x > 0, \quad \text{if} \quad \xi \neq 0, \\ \exp(-\exp(-x)), & x \in \mathbb{R}, \qquad\qquad \text{if} \quad \xi = 0, \end{cases} \tag{1.2}$$

the so-called general extreme-value (EV) distribution. If (1.1) holds, we are in the domain of attraction for maxima of $G_\xi$, with $\xi > 0$. This means that it is possible to linearly normalise the sequence of maximum values $\{X_{n:n} := \max(X_1, \ldots, X_n)\}_{n\geq 1}$, so that we get convergence to a non-degenerate random variable (r.v.) with d.f. $G_\xi$, in (1.2) (Gnedenko, 1943). We then write $F \in \mathcal{D}_{\mathcal{M}}(G_{\xi>0})$. This type of heavy-tailed models appears often in practice, in fields like telecommunication traffic (see Resnick, 1997, and Gomes, 2003), finance, insurance, economics, ecology (see Reiss and Thomas, 2001, 2007) and biometry (see Hüsler, 2009), among others. The parameter $\xi$, in (1.2), is the extreme-value index (EVI), one of the primary parameters of extreme events.

Let $F^{\leftarrow}$ denote the generalised inverse function of $F$, defined by $F^{\leftarrow}(t) := \inf\{x : F(x) \geq t\}$, and let $U$ be the reciprocal tail quantile function of the r.v. $X$, defined as

$$U(t) := F^{\leftarrow}(1 - 1/t), \; t \geq 1.$$

For heavy right tails, we assume the validity of any of the first-order conditions below:

$$F \in \mathcal{D}_{\mathcal{M}}(G_{\xi>0}) \quad \Longleftrightarrow \quad \overline{F} \in RV_{-1/\xi} \quad \Longleftrightarrow \quad U \in RV_\xi. \tag{1.3}$$

The second equivalence above was proved in de Haan (1984). For several technical proofs in the field of extreme value theory we further need information about the rate of convergence in (1.3), assuming that for every $x > 0$,

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \xi \ln x}{A(t)} = \psi_\rho(x) := \begin{cases} \frac{x^\rho - 1}{\rho}, & \text{if} \quad \rho < 0, \\ \ln x, & \text{if} \quad \rho = 0, \end{cases} \tag{1.4}$$

where $|A|$ must then be in $RV_\rho$ (Geluk and de Haan, 1987). Often, we further need information on the rate of convergence in (1.4), and assume that for all $x > 0$,

$$\lim_{t\to\infty} \frac{\frac{\ln U(tx) - \ln U(t) - \xi \ln x}{A(t)} - \psi_\rho(x)}{B(t)} = \begin{cases} \frac{x^{\rho+\rho'} - 1}{\rho+\rho'}, & \text{if} \quad \min(\rho, \rho') < 0, \\ \ln x, & \text{if} \quad \rho = \rho' = 0, \end{cases} \tag{1.5}$$

where $|B|$ must then be in $RV_{\rho'}$. Details on this precise third-order condition can be found in Gomes *et al.* (2002), Fraga Alves *et al.* (2003, 2006) and more generally in Wang and Cheng (2005).

For technical simplicity, we assume that $\rho < 0$ and that we can choose $A(t) = \xi\beta t^\rho$, in (1.4), with $\beta$ a non-null real number or even a slowly varying function, i.e. a regularly varying function with an index of regular variation equal to zero. This is equivalent to say that we are working with Pareto right tails such that for $C > 0$,

$$U(t) = Ct^\xi \bigl(1 + \xi\beta t^\rho/\rho + o(t^\rho)\bigr). \tag{1.6}$$

The pair of second-order parameters $(\beta, \rho)$, in (1.6), rules the rate of convergence in (1.4) and is dependent on a possible shift in the data. More precisely, if we have a location or shift parameter $s \in \mathbb{R}$, not necessarily null, i.e. if $F(x) = F_s(x) = F_0(x - s)$, then $U(t) \equiv U_s(t) = U_0(t) + s$ and also $(\beta, \rho) = (\beta_s, \rho_s)$ depend obviously on $s$, with

$$(\beta_s, \rho_s) := \begin{cases} (-s/C, -\xi), & \text{if} \quad \xi + \rho_0 < 0 \text{ and } s \neq 0, \\ (\beta_0 - s/C, \rho_0), & \text{if} \quad \xi + \rho_0 = 0 \text{ and } s \neq 0, \\ (\beta_0, \rho_0), & \qquad \text{otherwise}, \end{cases} \tag{1.7}$$

where $\beta_0$ and $\rho_0$ are respectively the scale and shape second-order parameters associated with an unshifted model ($s = 0$). Further details on the influence of a shift $s \neq 0$ in the second-order parameters are given in the Appendix.

The adequate estimation of the second-order parameters $\beta$ and $\rho$ is of primordial importance in the adaptive choice of the best number of top order statistics (o.s.'s) to be considered in the EVI-estimation, as well as in the construction of second-order reduced-bias (SORB) or minimum-variance reduced-bias (MVRB) EVI-estimators. Overviews of the subject can be found in Chapter 6 of the book by Reiss and Thomas (2007), Beirlant *et al.* (2012) and Gomes and Guillou (2014), among others. However, despite being scale invariant, many classes of EVI-estimators are not location-invariant. Since the EVI does not change with shifts, location invariance is surely a relevant and adequate property of the EVI-estimators. And such invariance can be attained through the use of the PORT-methodology, introduced in the sequel, with PORT standing for peaks over random thresholds, the terminology used in Araújo Santos *et al.* (2006).

Let $X_{i:n}$, $1 \le i \le n$, denote the o.s.'s associated with the random sample $\underline{X}_n = (X_1, \ldots, X_n)$ from an underlying d.f. $F$. The class of estimators suggested here is a function of the sample of excesses over a random threshold $X_{n_q:n}$, with $n_q = \lfloor nq \rfloor + 1$, where $\lfloor x \rfloor$ stands for the integer part of $x$. Such a sample is denoted by

$$\underline{X}_n^{(q)} := \left(X_{n:n} - X_{n_q:n}, X_{n-1:n} - X_{n_q:n}, \ldots, X_{n_q+1:n} - X_{n_q:n}\right), \tag{1.8}$$

where, we can have

- $0 < q < 1$, for any $F \in D_{\mathcal{M}}(G_{\xi>0})$ (the random threshold, $X_{n_q:n}$, is an empirical quantile);
- $q = 0$, for d.f.'s with a finite left endpoint $x_F := \inf\{x : F(x) > 0\}$ (the random threshold is the minimum, $X_{1:n}$).

Any statistical inference methodology based on the sample of excesses $\underline{X}_n^{(q)}$, defined in (1.8), is called a PORT-methodology. This methodology enabled the introduction and study of classical location/scale invariant EVI-estimators, like the PORT-Hill and the PORT-Moment estimators in Araújo Santos *et al.* (2006). These PORT EVI-estimators were further studied for finite-samples in Gomes *et al.* (2008). This methodology was also applied in the estimation of high quantiles in Henriques-Rodrigues and Gomes (2009). PORT MVRB EVI-estimators have been studied for finite samples and by Monte-Carlo simulation in Gomes *et al.* (2013), among others, and exhibit quite interesting features, being often possible to choose a value of $q$ that makes the sample path of these PORT MVRB EVI-estimators reasonably stable in $k$, making thus the choice of $k$ much more trivial than usual. All EVI-estimators are scale invariant, but there a few classes of scale and location invariant semi-parametric EVI-estimators, like Pickand's (Pickands, 1975), the peaks over threshold (Davison, 1984), and the 'pseudo' maximum likelihood EVI-estimators based on the excesses over an intermediate o.s, $X_{n-k:n}$, studied in Drees *et al.* (2004), and that can also be considered as PORT EVI-estimators.

The PORT methodology leads to location-invariant estimation, where the unshifted model $F_0$ plays thus a central role. In what follows, we use the notation $\chi_q$ for the $q$-quantile of the d.f. $F_0$, i.e. the value

$$\chi_q := F_0^{\leftarrow}(q) \tag{1.9}$$

(by convention $\chi_0 := x_{_F}$, the left endpoint of $F_0$). Since $n_q/n \to q$, as $n \to \infty$, we then know that the o.s. $X_{n_q:n}$, associated with a sample from $F_0$, is a consistent estimator for $F_0^{\leftarrow}(q)$ (van der Vaart, 1998, p.308), i.e. we have the following convergence in probability:

$$X_{n_q:n} \underset{n\to\infty}{\overset{p}{\longrightarrow}} \chi_q, \quad \text{for} \quad 0 \le q < 1 \quad \big(\chi_0 = x_{_F}\big). \tag{1.10}$$

When applying the PORT-methodology, we are working with the sample of excesses in (1.8), and we can assume that we are dealing with an unshifted d.f. $F_0$ underlying the r.v. $X_0$, to which we are inducing a random shift, strictly related to $\chi_q$, in (1.9). More precisely, we have a shift $s = -\chi_q$, i.e. we are working with $X_q := X_0 - \chi_q$, and use the simpler notation $(\beta_q, \rho_q)$ for $(\beta_{-\chi_q}, \rho_{-\chi_q})$, with $(\beta_s, \rho_s)$ defined in (1.7). Hence

$$(\beta_q, \rho_q) := \begin{cases} (\chi_q/C, -\xi), & \text{if} \quad \xi + \rho_0 < 0 \text{ and } \chi_q \neq 0, \\ (\beta_0 + \chi_q/C, \rho_0), & \text{if} \quad \xi + \rho_0 = 0 \text{ and } \chi_q \neq 0, \\ (\beta_0, \rho_0), & \qquad \text{otherwise.} \end{cases} \tag{1.11}$$

A class of location-invariant semi-parametric estimators of the so-called PORT-$\rho$ second-order parameter, $\rho_q$, in (1.11), was recently introduced and studied in Henriques-Rodrigues *et al.* (2014), among others. These authors mention that the main motivation for the theoretical study of a class of estimators of the shape second-order parameter $\rho_q$ is related to its possible use, concomitantly with a class of PORT estimators of the functional $A$, in (1.4), or at least of an adequate location-invariant estimator of the scale parameter of such a $A$-function, in the study of the asymptotic behaviour of second-order PORT-MVRB EVI-estimators. With the same motivation, we are now interested in the asymptotic behaviour of a class of location-invariant semi-parametric estimators of the so-called PORT-$\beta$ second-order parameter, $\beta_q$, also in (1.11). The technical contributions in Lemma 3.1 and Propositions 3.1 and 3.2 were already proved in the above mentioned article, and are crucial for the proofs in this paper. But the derivation of both consistency and asymptotic normality of the PORT-scale second-order parameter are quite technical and specific of the PORT-$\beta$ estimation. As mentioned above, our main and final objective is the incorporation of both results in the theoretical study of the PORT MVRB EVI-estimators, a work under progress. Such a study is obviously intricate but feasible, and out of the scope of this article.

In Section 2, we introduce the new class of PORT-$\beta$ estimators of the second-order parameter $\beta_q$ in (1.11). In Section 3, we present a few preliminary asymptotic results related to the PORT-methodology, under a third-order framework. In Section 4.1 we justify the class of PORT-$\beta$ estimators of the scale second-order parameter $\beta_q$, addressing the possibility of shifted heavy-tailed models, and refer the conditions required for their consistency. The non-degenerate asymptotic behaviour of the new class of estimators is presented in Section 4.2. In Section 5, we illustrate the finite sample behaviour of the new estimators through a Monte-Carlo simulation study. In Section 6, we present the proofs of the results stated in Section 4.1. Finally, in the Appendix we provide further details on the influence of a shift $s \neq 0$ in the second and third-order parameters.

## 2 The class of semi-parametric PORT-$\beta$ estimators

The building block of our estimators of the scale second-order parameter $\beta_q$, defined in (1.11) are the statistics used in Dekkers *et al.* (1989), Fraga Alves *et al.* (2003), Caeiro and Gomes (2006), and Henriques-Rodrigues *et al.* (2014), among others, i.e. for $\alpha > 0$ we consider the moment statistics of the log-excesses,

$$M_{n,k}^{(\alpha)} \equiv M_{n,k}^{(\alpha)}(\underline{X}_n) := \frac{1}{k}\sum_{i=1}^{k}\left(\ln X_{n-i+1:n} - \ln X_{n-k:n}\right)^{\alpha}, \tag{2.1}$$

but now applied to the sample of excesses $\underline{X}_n^{(q)}$, $0 \le q < 1$, in (1.8). For an intermediate $k$-sequence, i.e. a sequence $k = k_n$ of positive integers such that

$$k = k_n \to \infty \quad \text{and} \quad k = o(n) \quad \text{as} \quad n \to \infty, \tag{2.2}$$

we shall thus consider the location and scale-invariant statistics,

$$M_{n,k}^{(\alpha,q)} \equiv M_{n,k}^{(\alpha)}(\underline{X}_n^{(q)}) := \frac{1}{k}\sum_{i=1}^{k}\left(\ln\frac{X_{n-i+1:n} - X_{n_q:n}}{X_{n-k:n} - X_{n_q:n}}\right)^{\alpha}, \tag{2.3}$$

defined for $k < n - n_q$, with $M_{n,k}^{(\alpha)}(\underline{X}_n) \equiv M_{n,k}^{(\alpha)}$ given in (2.1), $\alpha > 0$.

Let $\mathbb{E}$ and $\mathbb{V}ar$ denote the mean value and variance operators, respectively, let $E$ denote a unit exponential r.v. and let $\Gamma(t)$ denote the complete Gamma function. For any real $\alpha > 0$, with $\xi \ge 0$ and $\rho < 0$, let us define

$$\begin{aligned}
\mu_\alpha^{(1)}(\xi) &:= \mathbb{E}\Big(E^\alpha e^{-\xi E}\Big) = \frac{\Gamma(\alpha+1)}{(1+\xi)^{\alpha+1}} \quad \mu_\alpha^{(1)} := \mu_\alpha^{(1)}(0) = \Gamma(\alpha+1), && (2.4)\\
\sigma_\alpha^{(1)} &:= \sqrt{\mathbb{V}\mathrm{ar}(E^\alpha)} = \sqrt{\Gamma(2\alpha+1) - \Gamma^2(\alpha+1)}, && (2.5)\\
\mu_\alpha^{(2)}(\xi,\rho) &:= \mathbb{E}\big(E^{\alpha-1}\, e^{-\xi E}\, (e^{\rho E} - 1)/\rho\big) = \frac{\Gamma(\alpha)}{\rho}\Big(\frac{(1+\xi)^\alpha - (1+\xi-\rho)^\alpha}{(1+\xi-\rho)^\alpha(1+\xi)^\alpha}\Big),\\
\mu_\alpha^{(2)}(\rho) &:= \mu_\alpha^{(2)}(0,\rho) = \frac{\Gamma(\alpha)}{\rho}\Big(\frac{1-(1-\rho)^\alpha}{(1-\rho)^\alpha}\Big),\\
\sigma_\alpha^{(2)}(\rho) &:= \sqrt{\mathbb{V}\mathrm{ar}\big(E^{\alpha-1}(e^{\rho E}-1)/\rho\big)} = \sqrt{\mu_{2\alpha}^{(3)}(\rho) - \big(\mu_\alpha^{(2)}(\rho)\big)^2},
\end{aligned}$$

$$\begin{aligned}
\eta_\alpha^{(3)}(\xi,\rho) &:= \mathbb{E}\Big(E^{\alpha-2}\,\Big((e^{-\xi E}-1)/(-\xi)\Big)\,\Big((e^{\rho E}-1)/\rho\Big)\Big)\\
&= \begin{cases} -\frac{1}{\xi\rho}\ln\frac{(1+\xi)(1-\rho)}{1+\xi-\rho}, & \text{if} \quad \alpha = 1\\ -\frac{\Gamma(\alpha)}{\xi\rho(\alpha-1)}\Big\{\frac{1}{(1+\xi-\rho)^{\alpha-1}} - \frac{1}{(1+\xi)^{\alpha-1}} - \frac{1}{(1-\rho)^{\alpha-1}} + 1\Big\}, & \text{if} \quad \alpha \ne 1,\end{cases}
\end{aligned}$$

and

$$\mu_\alpha^{(3)}(\rho) := \mathbb{E}\Big(E^{\alpha-2}\,\big((e^{\rho E}-1)/\rho\big)^2\Big) = \begin{cases} \frac{1}{\rho^2}\ln\frac{(1-\rho)^2}{1-2\rho}, & \text{if} \quad \alpha = 1\\ \frac{\Gamma(\alpha)}{\rho^2(\alpha-1)}\Big\{\frac{1}{(1-2\rho)^{\alpha-1}} - \frac{2}{(1-\rho)^{\alpha-1}} + 1\Big\}, & \text{if} \quad \alpha \ne 1.\end{cases}$$

Let us further introduce the notations:

$$\begin{aligned}
\overline{\mu}_\alpha^{(j)}(\rho) &:= \frac{\mu_\alpha^{(j)}(\rho)}{\mu_\alpha^{(1)}},\ j = 2,3, \qquad \overline{\mu}_\alpha^{(2)}(\xi,\rho) := \frac{\mu_\alpha^{(2)}(\xi,\rho)}{\mu_\alpha^{(1)}}, \quad \overline{\eta}_\alpha^{(3)}(\xi,\rho) := \frac{\eta_\alpha^{(3)}(\xi,\rho)}{\mu_\alpha^{(1)}}, && (2.6)\\
\overline{\sigma}_\alpha^{(1)} &:= \frac{\sigma_\alpha^{(1)}}{\mu_\alpha^{(1)}}, \qquad \overline{\sigma}_\alpha^{(2)}(\rho) := \frac{\sigma_\alpha^{(2)}(\rho)}{\mu_\alpha^{(1)}}, && (2.7)
\end{aligned}$$

and for any $\theta_1,\ \theta_2 > 0$,

$$d_{\alpha,\theta_1,\theta_2}(\rho) \ := \ \overline{\mu}^{(2)}_{\alpha\theta_1}(\rho) - \overline{\mu}^{(2)}_{\alpha\theta_2}(\rho). \tag{2.8}$$

For *tuning* parameters $\eta_q \in \mathbb{R}$ (as detected in Caeiro and Gomes, 2006), $\alpha, \theta_1, \theta_2 \in \mathbb{R}^+$, $\theta_1 \neq \theta_2$, we shall consider the PORT-versions of the r.v.'s used in the aforementioned paper for the estimation of $\beta$, in (1.6), i.e.

$$D^{(\alpha,\theta_1,\theta_2,\eta_q,q)}_{n,k}(\xi) := \left(\frac{M^{(\alpha\theta_1,q)}_{n,k}}{\mu^{(1)}_{\alpha\theta_1}}\right)^{\eta_q/\theta_1} - \left(\frac{M^{(\alpha\theta_2,q)}_{n,k}}{\mu^{(1)}_{\alpha\theta_2}}\right)^{\eta_q/\theta_2}, \tag{2.9}$$

with $M^{(\alpha,q)}_{n,k}$ and $\mu^{(1)}_\alpha$ defined in (2.3) and (2.4), respectively. As detailed in Section 4.1, under adequate conditions upon the growth of $k = k_n$, the study of the asymptotic behaviour of the r.v.'s $D^{(\alpha,\theta_1,\theta_2,\eta_q,q)}_{n,k}(\xi)$, in (2.9), enables us to introduce the class of consistent $\beta_q$-estimators, invariant for changes in location, and named PORT-$\beta$, given by

$$\widehat{\beta}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}_{n,k}(\widehat{\rho}^{(q)}) := \frac{2d_{2\alpha,\theta_1,\theta_2}(\widehat{\rho}^{(q)})}{\alpha\eta_q d^2_{\alpha,\theta_1,\theta_2}(\widehat{\rho}^{(q)})}\left(\frac{k}{n}\right)^{\widehat{\rho}^{(q)}} \frac{\left(D^{(\alpha,\theta_1,\theta_2,\eta_q,q)}_{n,k}(\xi)\right)^2}{D^{(2\alpha,\theta_1,\theta_2,\eta_q,q)}_{n,k}(\xi)}, \tag{2.10}$$

with tuning parameters $\alpha, \theta_1, \theta_2 > 0$, $\theta_1 \neq \theta_2$, $\eta_q \in \mathbb{R}$, $q \in [0, 1)$, $d_{\alpha,\theta_1,\theta_2}(\rho)$ and $D^{(\alpha,\theta_1,\theta_2,\eta_q,q)}_{n,k}(\xi)$ given in (2.8) and (2.9), respectively, and with $\widehat{\rho}^{(q)}$ the class of consistent $\rho_q$-estimators, invariant for changes in location, studied in Henriques-Rodrigues *et al.* (2014). The class of PORT-$\rho$ estimators of the shape second-order parameter $\rho_q$, similar to the simplest class of $\rho$-estimators in Fraga Alves *et al.* (2003), is also dependent on a tuning parameter $\tau_q \in \mathbb{R}$ and is given by

$$\widehat{\rho}^{(q)}_k \equiv \widehat{\rho}^{(\tau_q,q)}_k \equiv \widehat{\rho}^{(1,2,3,\tau_q,q)}_{n,k|T} := \frac{3\big(T^{(1,2,3,\tau_q,q)}_{n,k} - 1\big)}{T^{(1,2,3,\tau_q,q)}_{n,k} - 3}\mathbb{1}\big\{T^{(1,2,3,\tau_q,q)}_{n,k} \in (1,3)\big\}, \tag{2.11}$$

where $\mathbb{1}\{A\}$ denotes the indicator function of the event $A$, and with $M^{(\alpha,q)}_{n,k}$ given in (2.3),

$$T^{(1,2,3,\tau_q,q)}_{n,k} := \frac{\left(M^{(1,q)}_{n,k}\right)^{\tau_q} - \left(M^{(2,q)}_{n,k}/2\right)^{\tau_q/2}}{\left(M^{(2,q)}_{n,k}/2\right)^{\tau_q/2} - \left(M^{(3,q)}_{n,k}/6\right)^{\tau_q/3}},$$

for any $\tau_q \in \mathbb{R}$, with the notation $a^{b\tau} = b\ln a$ whenever we consider $\tau_q = 0$. Moreover, $\widehat{\rho}^{(q)} := \widehat{\rho}^{(q)}_{\lfloor (n-n_q)^{0.999}\rfloor}$. Note that for the estimation of second-order parameters, the choice of $k_n$ seems to be not crucial. A choice of the type $\lfloor n^{1-\epsilon}\rfloor$ usually works well both for the estimation of $\beta$ and $\rho$ provided that we choose tuning parameters $q$ and $\eta_q$ leading to high stability for large values of $k$.

Caeiro and Gomes (2006) suggest in practice the consideration of $(\alpha, \theta_1, \theta_2) = (1, 1, 2)$, for the classic $\beta$-estimation, in order to get a class of estimators dependent only on a tuning parameter $\eta \in \mathbb{R}$. Taking into account this suggestion we are led to the following functional expression for the PORT-$\beta$ estimators

$$\begin{aligned} \widehat{\beta}_k^{(\eta_q,q)} &\equiv \widehat{\beta}_{n,k}^{(1,1,2,\eta_q,q)}(\widehat{\rho}^{(q)}) \\ &:= \begin{cases} -\frac{2(2-\widehat{\rho}^{(q)})^2}{\eta_q \widehat{\rho}^{(q)}} \left(\frac{k}{n}\right)^{\widehat{\rho}^{(q)}} \frac{\left[\left(M_{n,k}^{(1,q)}\right)^{\eta_q} - \left(M_{n,k}^{(2,q)}/2\right)^{\eta_q/2}\right]^2}{\left(M_{n,k}^{(2,q)}/2\right)^{\eta_q} - \left(M_{n,k}^{(4,q)}/24\right)^{\eta_q/2}}, & \text{if} \quad \eta_q \neq 0, \\ -\frac{2(2-\widehat{\rho}^{(q)})^2}{\widehat{\rho}^{(q)}} \left(\frac{k}{n}\right)^{\widehat{\rho}^{(q)}} \frac{\left[\ln\left(M_{n,k}^{(1,q)}\right) - \frac{1}{2}\ln\left(M_{n,k}^{(2,q)}/2\right)\right]^2}{\ln\left(M_{n,k}^{(2,q)}/2\right) - \frac{1}{2}\ln\left(M_{n,k}^{(4,q)}/24\right)}, & \text{if} \quad \eta_q = 0. \end{cases} \end{aligned} \tag{2.12}$$

This new class of PORT-$\beta$ estimators depends on the tuning parameters $\eta_q \in \mathbb{R}$ and $q \in [0, 1)$, related to the PORT-methodology. These two tuning parameters provide an adequate flexible class of estimators of $\beta_q$, and their non-PORT versions, with a unique parameter, say $\eta \in \mathbb{R}$, have revealed to be suitable for practical purposes, despite of high volatile for small up to moderate $k$ comparatively to the $\beta$-estimators in Gomes and Martins (2002). The choice of the tuning parameter $q$ can be performed with a generalisation of the algorithm proposed in Gomes and Henriques-Rodrigues (2012). This research, is however, beyond of the scope of this paper, where a heuristic choice is provided.

# 3 Technical results related to the PORT-methodology

## 3.1 The second-order PORT-framework for heavy-tailed models

Under the aforementioned set-up in Section 1, the transformed r.v., $X_q = X_0 - \chi_q$, with $\chi_q$ given in (1.9), has an associated quantile function given by $U_q(t) = U_0(t) - \chi_q$. The second-order condition in (1.4) translates as

$$\lim_{t\to\infty} \frac{\ln U_q(tx) - \ln U_q(t) - \xi \ln x}{A_q(t)} = \psi_{\rho_q}(x) := \begin{cases} \frac{x^{\rho_q}-1}{\rho_q}, & \text{if} \quad \rho_q < 0, \\ \ln x, & \text{if} \quad \rho_q = 0, \end{cases} \tag{3.1}$$

for all $x > 0$. Moreover, $|A_q| \in RV_{\rho_q}$, $\rho_q \leq 0$, and $A_q$ relates to $A_0$ according to the following lemma.

**Lemma 3.1** (Henriques-Rodrigues *et al.*, 2014)**.** *Assume that $U_0 \in RV_\xi$ satisfies the second-order condition in* (3.1). *Then $U_q(t) = U_0(t) - \chi_q$, with $\chi_q$ defined in* (1.9), *is such that $U_q \in RV_\xi$ and* (3.1) *holds with $(\beta_q, \rho_q)$ given in* (1.11) *and*

$$A_q(t) := \begin{cases} \xi\chi_q/U_0(t), & \text{if } \ \xi+\rho_0<0 \ \textit{and} \ \chi_q \neq 0, \\ A_0(t)+\xi\chi_q/U_0(t), & \text{if } \ \xi+\rho_0=0 \ \textit{and} \ \chi_q \neq 0, \\ A_0(t), & \text{if } \ \xi+\rho_0>0 \ \textit{or} \ \chi_q = 0. \end{cases} \tag{3.2}$$

## 3.2 Third-order framework and asymptotic behaviour of auxiliary statistics

Next, we present the asymptotic behaviour of the statistics $M_{n,k}^{(\alpha,q)}$ defined in (2.3), based on the sample of excesses $\underline{X}_n^{(q)}$, $0 \le q < 1$, defined in (1.8). This requires a third-order framework because we further need to know the rate of convergence in (3.1). The third-order condition in (1.5) translates as

$$\lim_{t\to\infty} \frac{\frac{\ln U_q(tx)-\ln U_q(t)-\xi\ln x}{A_q(t)} - \psi_{\rho_q}(x)}{B_q(t)} = \begin{cases} \frac{x^{\rho_q+\rho'_q}-1}{\rho_q+\rho'_q}, & \text{if } \ \min(\rho_q,\rho'_q)<0, \\ \ln x, & \text{if } \ \rho_q=\rho'_q=0, \end{cases} \tag{3.3}$$

where $|B_q|$ must then be in $RV_{\rho'_q}$. For technical simplicity, we shall assume that $\rho_q,\ \rho'_q < 0$.

Let us further introduce the following notations. With $E_i$ independent and identically distributed (i.i.d.) unit exponential r.v.'s, and, with $\sigma_\alpha^{(1)}$ given in (2.5), define the asymptotically standard normal r.v.'s

$$Z_k^{(\alpha)} := \sqrt{k}\Big(\tfrac{1}{k}\sum_{i=1}^{k} E_i^\alpha - \Gamma(\alpha+1)\Big)/\sigma_\alpha^{(1)}.$$

Now, together with (2.7), we can combine these as follows:

$$W_k^{(\alpha,\theta_1,\theta_2)} := \overline{\sigma}_{\alpha\theta_1}^{(1)} Z_k^{(\alpha\theta_1)}/\theta_1 - \overline{\sigma}_{\alpha\theta_2}^{(1)} Z_k^{(\alpha\theta_2)}/\theta_2. \tag{3.4}$$

Finally, for $\eta \in \mathbb{R}$, $\alpha,\theta>0$, and with $\big(\overline{\mu}_\alpha^{(2)}(\rho), \overline{\mu}_\alpha^{(2)}(\xi,\rho), \overline{\eta}_\alpha^{(3)}(\xi,\rho)\big)$ defined in (2.6), we define

$$\begin{aligned} c_{\alpha,\theta,\eta}(\rho) &:= (\alpha\theta-1)\overline{\mu}_{\alpha\theta}^{(3)}(\rho) + \alpha(\eta-\theta)\big(\overline{\mu}_{\alpha\theta}^{(2)}(\rho)\big)^2, &(3.5)\\ g_{\alpha,\theta,\eta}(\xi,\rho) &:= \overline{\mu}_{\alpha\theta}^{(2)}(\xi,\rho) + (\alpha\theta-1)\overline{\eta}_{\alpha\theta}^{(3)}(\xi,\rho) + \alpha(\eta-\theta)\overline{\mu}_{\alpha\theta}^{(2)}(\rho)\overline{\mu}_{\alpha\theta}^{(2)}(-\xi), &(3.6)\\ h_{\alpha,\theta,\eta}(\xi) &:= 2\overline{\mu}_{\alpha\theta}^{(2)}(-2\xi) + (\alpha\theta-1)\overline{\mu}_{\alpha\theta}^{(3)}(-\xi) + \alpha(\eta-\theta)\Big(\overline{\mu}_{\alpha\theta}^{(2)}(-\xi)\Big)^2. &(3.7) \end{aligned}$$

Note that the statistics $M_{n,k}^{(\alpha,q)}$, in (2.3), depend on $q$ through $\chi_q$, in (1.9) (see also (1.10)), but are obviously independent on any shift $s$ imposed to the data. We can thus assume throughout that $s = 0$. We next present, under the third-order framework provided in (3.3), the asymptotic behaviour, as $n \to \infty$, of $M_{n,k}^{(\alpha,q)}$ and $D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}$, in (2.3) and (2.9), respectively, based on the sample of excesses $\underline{\mathbf{X}}_n^{(q)}$, $0 \le q < 1$, in (1.8). These results were stated and proven in Henriques-Rodrigues *et al.* (2014):

**Proposition 3.1** (Henriques-Rodrigues *et al.*, 2014)**.** *Let us assume that* (2.2) *holds, as well as the third-order condition in* (3.3)*, for* $\rho_0, \rho_0' < 0$. *We then get for* $M_{n,k}^{(\alpha,q)}$*, in* (2.3)*,* $\alpha > 0$*,* $k < n - n_q$*, with* $\chi_q$ *and* $M_{n,k}^{(\alpha)}$ *(for* $s = 0$*), given in* (1.10) *and* (2.1)*, respectively,* $\mu_\alpha^{(1)}$ *and* $\big(\overline{\mu}_\alpha^{(2)}(\rho), \overline{\mu}_\alpha^{(2)}(\xi,\rho), \overline{\mu}_\alpha^{(3)}(\rho), \overline{\eta}_\alpha^{(3)}(\xi,\rho)\big)$ *respectively given in* (2.4) *and* (2.6)*, the distributional representation,*

$$\begin{aligned} M_{n,k}^{(\alpha,q)} \;\stackrel{d}{=}\; M_{n,k}^{(\alpha)} + \tfrac{\alpha\xi^\alpha \mu_\alpha^{(1)} \chi_q}{U_0(n/k)} \Big\{ & \overline{\mu}_\alpha^{(2)}(-\xi) + \tfrac{\overline{\mu}_\alpha^{(2)}(\xi,\rho_0) + (\alpha-1)\ \overline{\eta}_\alpha^{(3)}(\xi,\rho_0)}{\xi}\ A_0(n/k)(1 + o_p(1)) \\ & + \tfrac{\chi_q}{U_0(n/k)} \Big( \overline{\mu}_\alpha^{(2)}(-2\xi) + \tfrac{(\alpha-1)}{2} \overline{\mu}_\alpha^{(3)}(-\xi) \Big)(1 + o_p(1)) \Big\}. \end{aligned}$$

Let us introduce the notations,

$$\begin{aligned} u_{\alpha,\theta_1,\theta_2,\eta}(\rho) &:= \big\{ c_{\alpha,\theta_1,\eta}(\rho) - c_{\alpha,\theta_2,\eta}(\rho) \big\}/(2\xi), && (3.8) \\ v_{\alpha,\theta_1,\theta_2}(\rho,\rho') &:= \overline{\mu}_{\alpha\theta_1}^{(2)}(\rho+\rho') - \overline{\mu}_{\alpha\theta_2}^{(2)}(\rho+\rho') \;\equiv\; d_{\alpha,\theta_1,\theta_2}(\rho+\rho'), && (3.9) \\ w_{\alpha,\theta_1,\theta_2,\eta}(\xi,\rho) &:= \{ g_{\alpha,\theta_1,\eta}(\xi,\rho) - g_{\alpha,\theta_2,\eta}(\xi,\rho) \}/\xi, && (3.10) \\ y_{\alpha,\theta_1,\theta_2,\eta}(\xi) &:= \{ h_{\alpha,\theta_1,\eta}(\xi) - h_{\alpha,\theta_2,\eta}(\xi) \}/2, && (3.11) \end{aligned}$$

with $d_{\alpha,\theta_1,\theta_2}(\rho)$, $c_{\alpha,\theta,\eta}(\rho)$, $g_{\alpha,\theta,\eta}(\xi,\rho)$ and $h_{\alpha,\theta,\eta}(\xi)$ defined in (2.8), (3.5), (3.6) and (3.7), respectively.

**Proposition 3.2** (Henriques-Rodrigues *et al.*, 2014)**.** *For intermediate* $k$*, as in* (2.2)*, let us assume the validity of the third-order condition in* (3.3)*. We then get for* $D_{n,k}^{(\alpha,q)}$*, in* (2.9)*,* $\alpha > 0$*,* $k < n - n_q$*, with* $\chi_q \neq 0$ *and* $d_{\alpha,\theta_1,\theta_2}(\rho)$ *given in* (1.10) *and* (2.8)*, respectively, the distributional* *representation,*

$$D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi) \stackrel{d}{=} \alpha\eta_q\xi^{\alpha\eta_q}\Bigg(\frac{W_k^{(\alpha,\theta_1,\theta_2)}}{\alpha\sqrt{k}} + \frac{A_0(n/k)}{\xi}\Big\{d_{\alpha,\theta_1,\theta_2}(\rho_0)$$

$$+ u_{\alpha,\theta_1,\theta_2,\eta_q}(\rho_0)A_0(n/k)(1+o_p(1)) + v_{\alpha,\theta_1,\theta_2}(\rho_0,\rho_0')B_0(n/k)(1+o_p(1))\Big\}$$

$$+ \frac{\chi_q}{U_0(n/k)}\Big\{d_{\alpha,\theta_1,\theta_2}(-\xi) + w_{\alpha,\theta_1,\theta_2,\eta_q}(\xi,\rho_0)A_0(n/k)(1+o_p(1)) + \frac{\chi_q\ y_{\alpha,\theta_1,\theta_2,\eta_q}(\xi)}{U_0(n/k)}(1+o_p(1))\Big\}\Bigg), \tag{3.12}$$

*where* $W_k^{(\alpha,\theta_1,\theta_2)}$ *is the asymptotic standard normal r.v. in* (3.4).

# 4 Asymptotic behaviour of the PORT-$\beta$ estimators

## 4.1 Consistency of the PORT-$\beta$ estimators

From the definition of the parameter $\beta_q$, in (1.11), we can see that the consistency of the PORT-$\beta$ estimators is related to the vector $(\xi, \rho_0)$ of the unshifted model $F_0$ associated with the available data. Therefore we shall consider three different regions:

(i) $\mathcal{R}_1 := \{\xi + \rho_0 < 0 \text{ and } \chi_q \neq 0\}$,

(ii) $\mathcal{R}_2 := \{\xi + \rho_0 > 0 \text{ or } (\xi + \rho_0 \leq 0 \text{ and } \chi_q = 0)\}$,

(iii) $\mathcal{R}_3 := \{\xi + \rho_0 = 0 \text{ and } \chi_q \neq 0\}$.

We may state the following:

**Theorem 4.1.** *Under the validity of the second-order condition in* (3.1), *with* $\rho_q < 0$, $(\beta_q, \rho_q)$ *defined in* (1.11), $\widehat{\rho}^{(q)}$ *any consistent estimator of* $\rho_q$ *such that* $(\widehat{\rho}^{(q)} - \rho_q)\ln(n/k) = o_p(1)$, *and with* $\widehat{\beta}_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}$ *defined in* (2.10),

$$\widehat{\beta}_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)} \underset{n\to\infty}{\overset{p}{\longrightarrow}} \beta_q,$$

*for any real* $\alpha > 0$, $\eta_q \in \mathbb{R}$, $\theta_1, \theta_2 \in \mathbb{R}^+\backslash\{1\}, \theta_1 \neq \theta_2$ *and* $0 < q < 1$ *or* $q = 0$ *if* $\chi_0$ *is finite, provided that* $k$ *is an intermediate sequence, as in* (2.2), *and moreover*

$$\sqrt{k}A_q(n/k) \to \infty,\ \text{as}\ n \to \infty, \tag{4.1}$$

*with* $A_q(\cdot)$ *defined in* (3.2).

**Remark 4.1.** *Note that when we consider models* $F_0 \in \mathcal{R}_1$, $A_0(t) = o(1/U_0(t))$ *and with* $A_q(t) = \xi\chi_q/U_0(t)$, *by* (3.2), *condition* (4.1) *corresponds to* $\sqrt{k}/U_0(n/k) \to \infty$, *as* $n \to \infty$. *For models* $F_0 \in \mathcal{R}_2$, $1/U_0(t) = o(A_0(t))$ *and since* $A_q(t) = A_0(t)$, *condition* (4.1) *is equivalent to* $\sqrt{k}A_0(n/k) \to \infty$, *as* $n \to \infty$. *Finally, for models* $F_0 \in \mathcal{R}_3$, $1/U_0(t) = O(A_0(t))$ *and since* $A_q(t) = A_0(t) + \xi\chi_q/U_0(t)$, *condition* (4.1) *is equivalent to* $\sqrt{k}A_0(n/k) \to \infty$ *or* $\sqrt{k}/U_0(n/k) \to \infty$, *as* $n \to \infty$.

## 4.2 Non-degenerate asymptotic behaviour of the PORT-$\beta$ estimators

In this section, and under a third-order framework, we derive the non-degenerate asymptotic properties of the PORT-$\beta$ classes of estimators introduced with all the generality in (2.10), and particularised in (2.12). We first state the following result:

**Proposition 4.1** (Fraga Alves *et al.*, 2003). *Under the validity of the second-order condition in* (1.4), *with* $\rho < 0$, *if* (2.2) *holds and* $\sqrt{k}A(n/k) \to \infty$, *as* $n \to \infty$, *the asymptotic variance of* $W_k^{(\alpha,\theta_1,\theta_2)}$, *in* (3.4), *is*

$$\sigma^2_{W|\alpha,\theta_1,\theta_2} = \frac{2}{\alpha}\left(\frac{\Gamma(2\alpha\theta_1)}{\theta_1^3\Gamma^2(\alpha\theta_1)} + \frac{\Gamma(2\alpha\theta_2)}{\theta_2^3\Gamma^2(\alpha\theta_2)} - \frac{(\theta_1+\theta_2)\Gamma(\alpha(\theta_1+\theta_2))}{\theta_1^2\theta_2^2\Gamma(\alpha\theta_1)\Gamma(\alpha\theta_2)}\right) - \left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)^2. \tag{4.2}$$

**Proposition 4.2** (Caeiro and Gomes, 2006). *Under the validity of the second-order condition in* (1.4), *with* $\rho < 0$, *if* (2.2) *holds and* $\sqrt{k}A(n/k) \to \infty$, *as* $n \to \infty$, *the asymptotic covariance of* $(W_k^{(\alpha,\theta_1,\theta_2)}, W_k^{(2\alpha,\theta_1,\theta_2)})$, *with* $W_k^{(\alpha,\theta_1,\theta_2)}$ *in* (3.4), *is given by*

$$\begin{aligned}\overline{\sigma}_{W|\alpha,\theta_1,\theta_2} = \frac{1}{2\alpha}\Bigg(&\frac{3\Gamma(3\alpha\theta_1)}{\theta_1^3\Gamma(\alpha\theta_1)\Gamma(2\alpha\theta_1)} - \frac{(\theta_1+2\theta_2)\Gamma(\alpha(\theta_1+2\theta_2))}{\theta_1^2\theta_2^2\Gamma(\alpha\theta_1)\Gamma(2\alpha\theta_2)} \\ &- \frac{(2\theta_1+\theta_2)\Gamma(\alpha(2\theta_1+\theta_2))}{\theta_1^2\theta_2^2\Gamma(2\alpha\theta_1)\Gamma(\alpha\theta_2)} + \frac{3\Gamma(3\alpha\theta_2)}{\theta_2^3\Gamma(\alpha\theta_2)\Gamma(2\alpha\theta_2)}\Bigg) - \left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)^2.\end{aligned} \tag{4.3}$$

Let us further use the notations:

$$y^{(\alpha,\theta_1,\theta_2,\eta_q)}(\xi,-\xi) := \frac{2y_{\alpha,\theta_1,\theta_2,\eta_q}(\xi)}{d_{\alpha,\theta_1,\theta_2}(-\xi)} - \frac{y_{2\alpha,\theta_1,\theta_2,\eta_q}(\xi)}{d_{2\alpha,\theta_1,\theta_2}(-\xi)}, \tag{4.4}$$

$$z^{(\alpha,\theta_1,\theta_2)}(\rho,-\xi) := \frac{2d_{\alpha,\theta_1,\theta_2}(\rho)}{d_{\alpha,\theta_1,\theta_2}(-\xi)} - \frac{d_{2\alpha,\theta_1,\theta_2}(\rho)}{d_{2\alpha,\theta_1,\theta_2}(-\xi)}, \tag{4.5}$$

$$u^{(\alpha,\theta_1,\theta_2,\eta_q)}(\rho) := \frac{2u_{\alpha,\theta_1,\theta_2,\eta_q}(\rho)}{d_{\alpha,\theta_1,\theta_2}(\rho)} - \frac{u_{2\alpha,\theta_1,\theta_2,\eta_q}(\rho)}{d_{2\alpha,\theta_1,\theta_2}(\rho)}, \tag{4.6}$$

$$v^{(\alpha,\theta_1,\theta_2)}(\rho,\rho') := \frac{2v_{\alpha,\theta_1,\theta_2}(\rho,\rho')}{d_{\alpha,\theta_1,\theta_2}(\rho)} - \frac{v_{2\alpha,\theta_1,\theta_2}(\rho,\rho')}{d_{2\alpha,\theta_1,\theta_2}(\rho)}, \tag{4.7}$$

$$w^{(\alpha,\theta_1,\theta_2,\eta_q)}(\xi,\rho) := \frac{2w_{\alpha,\theta_1,\theta_2,\eta_q}(\xi,\rho)}{d_{\alpha,\theta_1,\theta_2}(\rho)} - \frac{w_{2\alpha,\theta_1,\theta_2,\eta_q}(\xi,\rho)}{d_{2\alpha,\theta_1,\theta_2}(\rho)}, \tag{4.8}$$

with $u_{\alpha,\theta_1,\theta_2,\eta}(\rho)$, $v_{\alpha,\theta_1,\theta_2,\eta}(\rho,\rho')$, $w_{\alpha,\theta_1,\theta_2,\eta}(\xi,\rho)$ and $y_{\alpha,\theta_1,\theta_2,\eta}(\xi)$ given in (3.8), (3.9), (3.10) and (3.11), respectively.

We can now state the non-degenerate asymptotic behaviour of the class of PORT-$\beta$ estimators, in (2.10). The regions $\mathcal{R}_1$ / $\mathcal{R}_2$ given above, are now split in

- $\mathcal{R}_{11} := \{\rho_0 < -2\xi$ and $\chi_q \neq 0\}$ / $\mathcal{R}_{21} := \{-\xi < \rho_0 < -\frac{\xi}{2}$ and $\chi_q \neq 0\}$,
- $\mathcal{R}_{12} := \{\rho_0 = -2\xi$ and $\chi_q \neq 0\}$ / $\mathcal{R}_{22} := \{\rho_0 = -\frac{\xi}{2}$ and $\chi_q \neq 0\}$, and
- $\mathcal{R}_{13} := \{-2\xi < \rho_0 < -\xi$ and $\chi_q \neq 0\}$ / $\mathcal{R}_{23} := \{\frac{\xi}{2} < \rho_0 < 0$ or $(\xi > -\rho_0$ and $\chi_q = 0)\}$.

**Theorem 4.2.** *Let us assume that the third-order condition in* (3.3) *holds, with $\rho_0$, $\rho_0' < 0$ and consider the* PORT-$\rho$ *class of estimators, $\widehat{\beta}_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}$, defined in* (2.10)*, with $\beta_q$ given in* (1.11)*. Then, with $\theta_1 < \theta_2$, real numbers different from* 1*, $\alpha > 0$, $\eta_q \in \mathbb{R}$ and $0 < q < 1$ or $q = 0$ provided that $\chi_0 = x_F$ is finite, and intermediate sequences of positive integers $k = k_n$, as in* (2.2)*, such that* (4.1) *holds.*

i) *In $\mathcal{R}_1$, if we further assume that $\lim_{n\to\infty} \sqrt{k}A_0(n/k) = \lambda$ and $\lim_{n\to\infty} \sqrt{k}/U_0^2(n/k) = \lambda_U$, we get*

$$\frac{\sqrt{k}}{U_0(n/k)}\left(\widehat{\beta}_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\rho_q) - \beta_q\right) \xrightarrow[n\to\infty]{d} \mathcal{N}\left(\overset{\bullet}{\mu}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}, \overset{\bullet}{\sigma}{}^2_{\alpha,\theta_1,\theta_2,q}\right),$$

*with*

$$\overset{\bullet}{\mu}^{(\alpha,\theta_1,\theta_2,\eta_q,q)} = \begin{cases} \beta_q\chi_q\ \lambda_U y^{(\alpha,\theta_1,\theta_2,\eta_q)}(\xi,-\xi), & \text{in } \mathcal{R}_{11} \\ \beta_q\left(\frac{\lambda\ z^{(\alpha,\theta_1,\theta_2)}(\rho_0,-\xi)}{\xi\chi_q} + \chi_q\ \lambda_U y^{(\alpha,\theta_1,\theta_2,\eta_q)}(\xi,-\xi)\right), & \text{in } \mathcal{R}_{12} \\ \beta_q\frac{\lambda\ z^{(\alpha,\theta_1,\theta_2)}(\rho_0,-\xi)}{\xi\chi_q}, & \text{in } \mathcal{R}_{13}, \end{cases}$$

*$y^{(\alpha,\theta_1,\theta_2,\eta)}(\xi,\rho)$ and $z^{(\alpha,\theta_1,\theta_2)}(\rho,-\xi)$ defined in* (4.4) *and* (4.5)*, respectively. Moreover,*

$$\begin{aligned}\overset{\bullet}{\sigma}{}^2_{\alpha,\theta_1,\theta_2,q} &= \left(\frac{\beta_q}{\alpha\chi_q}\right)^2 \mathbb{V}ar\left(\frac{2W_k^{(\alpha,\theta_1,\theta_2)}}{d_{\alpha,\theta_1,\theta_2}(-\xi)} - \frac{W_k^{(2\alpha,\theta_1,\theta_2)}}{2d_{2\alpha,\theta_1,\theta_2}(-\xi)}\right) \\ &= \frac{1}{(C\alpha)^2}\left[\frac{4\sigma^2_{W|\alpha,\theta_1,\theta_2}}{d^2_{\alpha,\theta_1,\theta_2}(-\xi)} + \frac{\sigma^2_{W|2\alpha,\theta_1,\theta_2}}{4d^2_{2\alpha,\theta_1,\theta_2}(-\xi)} - \frac{2\overline{\sigma}_{W|\alpha,\theta_1,\theta_2}}{d_{\alpha,\theta_1,\theta_2}(-\xi)d_{\alpha,\theta_1,\theta_2}(-\xi)}\right],\end{aligned}$$

*with $\sigma^2_{W|\alpha,\theta_1,\theta_2}$, $\overline{\sigma}_{W|\alpha,\theta_1,\theta_2}$ given in* (4.2), (4.3)*, respectively.*

ii) *In $\mathcal{R}_2$, if we further assume that $\lim_{n\to\infty} \sqrt{k}A_0^2(n/k) = \lambda_A$, $\lim_{n\to\infty} \sqrt{k}A_0(n/k)B_0(n/k) = \lambda_B$ and $\lim_{n\to\infty} \sqrt{k}/U_0(n/k) = \lambda'$, we get*

$$\sqrt{k}A_0(n/k)\left(\widehat{\beta}_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)} - \beta_q\right) \xrightarrow[n\to\infty]{d} \mathcal{N}\left(\mu^{(\alpha,\theta_1,\theta_2,\eta_q,q)}, \sigma^2_{\alpha,\theta_1,\theta_2,q}\right),$$

*where with* $\mu^{(\alpha,\theta_1,\theta_2,\eta_q)}(\rho_0,\rho_0') := u^{(\alpha,\theta_1,\theta_2,\eta_q)}(\rho_0)\lambda_A + v^{(\alpha,\theta_1,\theta_2)}(\rho_0,\rho_0')\lambda_B$, *and* $z^{(\alpha,\theta_1,\theta_2,\eta)}(-\xi,\rho)$, $u^{(\alpha,\theta_1,\theta_2,\eta)}(\rho)$ *and* $v^{(\alpha,\theta_1,\theta_2)}(\rho,\rho')$ *given in* (4.5), (4.6) *and* (4.7), *respectively,*

$$\mu^{(\alpha,\theta_1,\theta_2,\eta_q,q)} = \begin{cases} \beta_0\xi\chi_q\lambda' z^{(\alpha,\theta_1,\theta_2,\eta_q)}(-\xi,\rho_0), & \text{in } \mathcal{R}_{21} \\ \beta_0\big(\mu^{(\alpha,\theta_1,\theta_2,\eta_q)}(\rho_0,\rho_0') + \xi\chi_q\lambda' z^{(\alpha,\theta_1,\theta_2,\eta_q)}(-\xi,\rho_0)\big), & \text{in } \mathcal{R}_{22} \\ \beta_0\mu^{(\alpha,\theta_1,\theta_2,\eta_q)}, & \text{in } \mathcal{R}_{23}. \end{cases}$$

*Additionally,*

$$\begin{aligned}\sigma^2_{\alpha,\theta_1,\theta_2,q} &= \left(\tfrac{\xi\beta_q}{\alpha}\right)^2 \mathbb{V}ar\left(\frac{2W_k^{(\alpha,\theta_1,\theta_2)}}{d_{\alpha,\theta_1,\theta_2}(\rho_0)} - \frac{W_k^{(2\alpha,\theta_1,\theta_2)}}{2d_{2\alpha,\theta_1,\theta_2}(\rho_0)}\right) \\ &= \left(\tfrac{\xi\beta_0}{\alpha}\right)^2 \left[\frac{4\sigma^2_{W|\alpha,\theta_1,\theta_2}}{d^2_{\alpha,\theta_1,\theta_2}(\rho_0)} + \frac{\sigma^2_{W|2\alpha,\theta_1,\theta_2}}{4d^2_{2\alpha,\theta_1,\theta_2}(\rho_0)} - \frac{2\overline{\sigma}_{W|\alpha,\theta_1,\theta_2}}{d_{\alpha,\theta_1,\theta_2}(\rho_0)d_{\alpha,\theta_1,\theta_2}(\rho_0)}\right] =: \sigma^2_{\alpha,\theta_1,\theta_2},\end{aligned}$$

*with* $\sigma^2_{\alpha,\theta_1,\theta_2}$ *the asymptotic variance of the classical* $\beta$*-estimator introduced in* Caeiro and Gomes (2006), *and with* $\sigma^2_{W|\alpha,\theta_1,\theta_2}$, $\overline{\sigma}_{W|\alpha,\theta_1,\theta_2}$ *given in* (4.2), (4.3), *respectively.*

iii) *In* $\mathcal{R}_3$, *if we further assume that* $\lim_{n\to\infty}\sqrt{k}A_0^2(n/k) = \lambda_A$, $\lim_{n\to\infty}\sqrt{k}A_0(n/k)B_0(n/k) = \lambda_B$ *and* $\lim_{n\to\infty}\sqrt{k}A_0(n/k)/U_0(n/k) = \lambda_{AU}$ *and* $\widetilde{\lambda} = \lim_{n\to\infty} 1/\big(A_0(n/k)U_0(n/k)\big) \neq 0$, *we get*

$$\sqrt{k}A_0(n/k)\left(\widehat{\beta}_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)} - \beta_q\right) \xrightarrow[n\to\infty]{d} \mathcal{N}\left(\widetilde{\mu}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}, \widetilde{\sigma}^2_{\alpha,\theta_1,\theta_2,q}\right),$$

*with*

$$\widetilde{\sigma}^2_{\alpha,\theta_1,\theta_2,q} = \left(\frac{\beta_q\ \sigma_{\alpha,\theta_1,\theta_2}}{\beta_0(1+\xi\widetilde{\lambda}\chi_q)}\right)^2,$$

*where* $\sigma^2_{\alpha,\theta_1,\theta_2}$ *is the asymptotic variance of the classical* $\beta$*-estimator introduced in* Caeiro and Gomes (2006), *and*

$$\begin{aligned}\widetilde{\mu}^{(\alpha,\theta_1,\theta_2,\eta_q,q)} = \frac{\beta_q}{1+\xi\widetilde{\lambda}\chi_q}\Big(&u^{(\alpha,\theta_1,\theta_2,\eta_q)}(\rho_0)\lambda_A + v^{(\alpha,\theta_1,\theta_2)}(\rho_0,\rho_0')\lambda_B \\ &+ \xi\chi_q\left(w^{(\alpha,\theta_1,\theta_2,\eta_q)}(\xi,\rho_0) + \chi_q\widetilde{\lambda}\ y(\alpha,\theta_1,\theta_2,\eta_q)(\xi,\rho_0)\right)\lambda_{AU}\Big),\end{aligned}$$

*with* $y^{(\alpha,\theta_1,\theta_2,\eta_q)}(\xi,\rho)$, $u^{(\alpha,\theta_1,\theta_2,\eta)}(\rho)$, $v^{(\alpha,\theta_1,\theta_2)}(\rho,\rho')$ *and* $w^{(\alpha,\theta_1,\theta_2,\eta_q)}(\xi,\rho)$ *defined in* (4.4), (4.6), (4.7) *and* (4.8), *respectively.*

We finally present the non-degenerate behaviour of the PORT-$\beta$ estimators, in (2.12), dependent only on two *tuning parameters*, $(q, \eta_q)$. This is a particular case that can be relevant in practice, and where we get simpler expressions for asymptotic variances and bias.

**Corollary 4.1.** *Under the validity of the third-order condition in* (3.3)*, with* $\rho_0,\ \rho_0' < 0$*, and for the particular case* $(\alpha, \theta_1, \theta_2) = (1, 1, 2)$*, we have the validity of the following asymptotic distributional representation for the* PORT-$\beta$ *estimator,* $\widehat{\beta}_k^{(\eta_q,q)}$*, in* (2.12)*.*

i) *In* $\mathcal{R}_1$*, with* $\beta_q = \chi_q/C$*, and with the same notation as before for* $\mathcal{R}_{11}$*,* $\mathcal{R}_{12}$ *and* $\mathcal{R}_{13}$*,*

$$\begin{aligned}\widehat{\beta}_k^{(\eta_q,q)} &\stackrel{d}{=} \beta_q + \frac{\beta_q \overset{\bullet}{\sigma}_{\rho_q}}{\sqrt{k}/U_0(n/k)} W_k^{R_1} \\ &+ \begin{cases} \frac{\beta_q \chi_q\ y(\xi,-\xi)}{U_0(n/k)}(1+o_p(1)), & \text{in } \mathcal{R}_{11} \\ \beta_q \left(\frac{z(\rho_0,-\xi)}{\xi\chi_q} A_0(n/k)U_0(n/k) + \frac{\chi_q\ y(\xi,-\xi)}{U_0(n/k)}\right)(1+o_p(1)), & \text{in } \mathcal{R}_{12} \\ \beta_q \frac{z(\rho_0,-\xi)}{\xi\chi_q} A_0(n/k)U_0(n/k)(1+o_p(1)), & \text{in } \mathcal{R}_{13}, \end{cases}\end{aligned}$$

*where* $W_k^{R_1}$ *is asymptotically standard normal, and with*

$$A_\xi := 32\xi^8 + 212\xi^7 + 568\xi^6 + 780\xi^5 + 538\xi^4 + 93\xi^3 - 108\xi^2 - 68\xi - 12 \tag{4.9}$$

$$B_\xi := 6 + \xi(4+\xi) \tag{4.10}$$

$$C_{\xi,\rho} := \rho^3(21 + 5\xi B_\xi) - 2\rho^2(30 + 7\xi B_\xi) + 6\rho(9 + 2\xi B_\xi) - 2(6 + \xi B_\xi) \tag{4.11}$$

$$\begin{aligned} y(\xi,-\xi) &= \frac{\xi(2+\xi)(8+18\xi+16\xi^2+5\xi^3)(1+2\xi)^4\eta + 2\xi^2 A_\xi}{2(1+\xi)^2(2+\xi)^2(1+2\xi)^4(2+2\xi+\xi^2)} \\ z(\rho_0,-\xi) &= -\frac{(1+\xi)^5(2-\rho_0)C_{\xi,\rho_0}}{(2+\xi)^2(2+2\xi+\xi^2)(3+3\xi+\xi^2)(1-\rho_0)^6} \end{aligned}$$

*and* $\overset{\bullet}{\sigma}{}^2_{\rho_q} \equiv \overset{\bullet}{\sigma}{}^2_{-\xi} = \left(\frac{1}{\chi_q}\right)^2 \left(\frac{1+\xi}{2+\xi}\right)^2 \left(\frac{21\xi^4+68\xi^3+86\xi^2+68\xi+33}{\xi^2}\right)$.

ii) *In* $\mathcal{R}_2$*, with* $\beta_q \equiv \beta_0$ *and again with the same notation as before for* $\mathcal{R}_{21}$*,* $\mathcal{R}_{22}$ *and* $\mathcal{R}_{23}$*,*

$$\begin{aligned}\widehat{\beta}_k^{(\eta_q,q)} &\stackrel{d}{=} \beta_q + \frac{\beta_q \sigma_{\rho_q}}{\sqrt{k}A_0(n/k)} W_k^{R_2} \\ &+ \begin{cases} \beta_q \frac{\xi\chi_q z(-\xi,\rho_0)}{A_0(n/k)U_0(n/k)}(1+o_p(1)), & \text{in } \mathcal{R}_{21} \\ \beta_q \left(\mu(\rho_0,\rho_0') + \frac{\xi\chi_q z(-\xi,\rho_0)}{A_0(n/k)U_0(n/k)}\right)(1+o_p(1)), & \text{in } \mathcal{R}_{22} \\ \beta_q \mu(\rho_0,\rho_0')(1+o_p(1)), & \text{in } \mathcal{R}_{23}, \end{cases}\end{aligned}$$

*where* $\mu(\rho,\rho') = u_\rho A_0(n/k) + v_{\rho,\rho'} B_0(n/k)$*, with* $u_\rho$ *and* $v_{\rho,\rho'}$*, given by*

$$u_\rho \equiv u_\rho(\eta_q) = -\frac{2\rho^2\Big(4-\rho(2-\rho)\Big(16\rho^5-68\rho^4+116\rho^3-96\rho^2+33\rho+2\Big)\Big)}{2\xi(2-\rho)^2(1-\rho)^2(1-2\rho)^3(2-2\rho+\rho^2)} - \frac{\eta_q(2-\rho)(1-2\rho)^3\Big(5\rho^3-16\rho^2+18\rho-8\Big)}{2\xi(2-\rho)^2(1-\rho)^2(1-2\rho)^3(2-2\rho+\rho^2)} \quad (4.12)$$

*and*

$$v_{\rho,\rho'} = \frac{(1-\rho)^3\Big((2-\rho)^2(1-\rho)^3(2-2\rho+\rho^2)-2(2-\rho)^2(1-\rho)^2(5-4\rho+2\rho^2)\rho'\Big)}{(2-\rho)^2(2-2\rho+\rho^2)(1-(\rho+\rho'))^6} \frac{(1-\rho)^3\Big(2(1-\rho)(29-55\rho+44\rho^2-18\rho^3+3\rho^4)\rho'^2-2(17-35\rho+29\rho^2-12\rho^3+2\rho^4)\rho'^3\Big)}{(2-\rho)^2(2-2\rho+\rho^2)(1-(\rho+\rho'))^6} - \frac{(1-\rho)^3(-7+9\rho-5\rho^2+\rho^3)\rho'^4}{(2-\rho)^2(2-2\rho+\rho^2)(1-(\rho+\rho'))^6}, \quad (4.13)$$

*respectively. Moreover,* $W_k^{R_2}$ *is asymptotically standard normal,*

$$\begin{aligned} \sigma^2_{\rho_q} &\equiv \sigma^2_{\rho_0} \equiv \sigma^2 = \left(\frac{\xi(1-\rho_0)}{2-\rho_0}\right)^2 \left(\frac{21\rho_0^4-68\rho_0^3+86\rho_0^2-68\rho_0+33}{\rho_0^2}\right), \quad (4.14)\\ z(-\xi,\rho_0) &= -\frac{(1-\rho_0)^5(2+\xi)C_{-\rho_0,\xi}}{(2-\rho_0)^2(2-2\rho_0+\rho_0^2)(3-3\rho_0+\rho_0^2)(1+\xi)^6}, \end{aligned}$$

*with* $B_\xi$ *and* $C_{\xi,\rho}$ *defined in* (4.10) *and* (4.11)*, respectively.*

iii) *In* $\mathcal{R}_3$*, with* $\beta_q = \beta_0 + \chi_q/C$ *and with* $\widetilde{\lambda} = \lim_{n\to\infty} 1/\big(A_0(n/k)U_0(n/k)\big) = (\xi\beta_0 C)^{-1} \neq 0$*, with* $C$ *given in* (1.6)*,*

$$\widehat{\beta}_k^{(\eta_q,q)} \overset{d}{=} \beta_q + \frac{\beta_q\widetilde{\sigma}_{\rho_q}}{\sqrt{k}A_0(n/k)} W_k^{R_3} + \frac{\beta_q}{1+\xi\widetilde{\lambda}\chi_q}\big(u(\rho_0)A_0(n/k)+v(\rho_0,\rho_0')B_0(n/k)\big)(1+o_p(1)) + \frac{\beta_q}{1+\xi\widetilde{\lambda}\chi_q}\left(\frac{\xi\chi_q\Big(w(\xi,\rho_0)+\chi_q\widetilde{\lambda}\ y(\xi,\rho_0)\Big)}{U_0(n/k)}\right)(1+o_p(1)),$$

*where* $W_k^{R_3}$ *is an asymptotically standard normal r.v. with* $\widetilde{\sigma}^2_{\rho_q} = \Big(\sigma/(1+\xi\widetilde{\lambda}\chi_q)\Big)^2$*,* $\sigma^2$ *given in* (4.14)*,* $u_\rho$ *and* $v_{\rho,\rho'}$*, defined in* (4.12) *and* (4.13)*, respectively, and*

$$\begin{aligned} w(\xi,\rho_0) = w(-\rho_0,\rho_0) &= -\frac{(-8+18\rho_0-16\rho_0^2+5\rho_0^3)\eta_q}{(2-\rho_0)(1-\rho_0)^2(2-2\rho_0+\rho_0^2)} \\ &\quad + \frac{2\rho_0\Big(14-\rho_0(91-221\rho_0+216\rho_0^2+25\rho_0^3-252\rho_0^4+240\rho_0^5-100\rho_0^6+16\rho_0^7)\Big)}{(1-2\rho_0)^4(2-\rho_0)^2(1-\rho_0)^2(2-2\rho_0+\rho_0^2)}, \\ y(\xi,\rho_0) = y(-\rho_0,\rho_0) &= \frac{-\rho_0(2-\rho_0)(8-18\rho_0+16\rho_0^2-5\rho_0^3)(1-2\rho_0)^4\eta_q+2\rho_0^2A_{-\rho_0}}{2(1-\rho_0)^2(2-\rho_0)^2(1-2\rho_0)^4(2-2\rho_0+\rho_0^2))}, \end{aligned}$$

*with* $A_\xi$ *defined in* (4.9).

## 5 A Monte-Carlo simulation

As an illustration, we next present in Figures 1 to 4, the mean values (E) and the root mean square errors (RMSE), of the classical estimator, denoted by $\widehat{\beta}_k^{(\eta)}$, and the PORT-$\beta$ estimators $\left\{\widehat{\beta}_k^{(\eta_q,q)} \equiv \widehat{\beta}_{n,k}^{(1,1,2,\eta_q,q)}(\widehat{\rho}^{(q)})\right\}_{q=0,\ 0.1,\ 0.25}$, as defined in (2.12), as a function of the sample fraction $k/n$, for a sample size $n = 5000$. The results are associated with the output of a simulation of size 1000, related to underlying Fréchet parents $F_0(x) = \exp(-x^{-1/\xi}),\ x > 0$, with $\xi = 0.5$ and $\xi = 2$ $(\rho_0 = -1,\ \beta_0 = 0.5)$, Burr parents $F_0(x) = 1 - (1 + x^{-\rho/\xi})^{1/\rho},\ x > 0$ with $(\xi, \rho_0) = (0.5, -1)$, $(\xi, \rho_0) = (1, -0.5)$, $\beta_0 = 1$ and the shifted models $F_s(x) = F_0(x - s)$, with $s = 1$. We have here used $\tau_q = 0$, the value suggested in several other research papers for the PORT-$\rho$ estimators $\widehat{\rho}_k^{(q)} \equiv \widehat{\rho}_k^{(\tau_q,q)}$, in (2.11). The choice $\tau = 0$ has been heuristically suggested and used before for the classic $\rho$-estimation in the region $|\rho| \le 1$ (see Fraga Alves *et al.* (2003), for further details). The heuristic criteria proposed in Fraga Alves *et al.* (2003) also suggests the use of $\tau = 1$ in the region $|\rho| > 1$. And indeed $\tau$ can be even negative, as detected in Caeiro and Gomes (2006). Note that for both models we have $\xi + \rho_0 \neq 0$, and taking into account Remark 6.3 we have for the Fréchet model,

$$\beta_q \equiv \beta_q^F = \begin{cases} \chi_q = (-\ln q)^{-\xi}, & \text{if} \quad \xi + \rho_0 < 0 \text{ and } \chi_q \neq 0 \ (0 < q < 1), \\ \beta_0 = 0.5, & \text{if} \quad \xi + \rho_0 > 0 \text{ or } (\xi + \gamma < 0 \text{ and } \chi_q = 0 \ (q = 0)), \end{cases} \tag{5.1}$$

and for the Burr model

$$\beta_q \equiv \beta_q^B = \begin{cases} \chi_q = (q^{-1} - 1)^{-0.5}, & \text{if} \quad \xi + \rho_0 < 0 \text{ and } \chi_q \neq 0 \ (0 < q < 1), \\ \beta_0 = 1, & \text{if} \quad \xi + \rho_0 > 0 \text{ or } (\xi + \gamma < 0 \text{ and } \chi_q = 0 \ (q = 0)). \end{cases} \tag{5.2}$$

As mentioned in Caeiro and Gomes (2006) the choice of the tuning parameter $\eta$ related to classical $\beta$-estimator depends heavily on the model and the best choices for $\eta$ for the Fréchet and Burr models are provided by negative values of this control parameter. Taking this in consideration we propose the following heuristic criteria for the choice of the tuning parameter $\eta_q$ related to the class of estimators in (2.12). On the basis of the choice for $\eta_0$ provided in Caeiro and Gomes (2006), for models with a support $[0, \infty)$, $C = 1$ and a choice $0 \le q_1 < q_2 < q_r$

$(0 \leq \chi_{q_1} \leq \cdots \leq \chi_{q_r})$, choose:

$$\eta_q = \begin{cases} \frac{\eta_0}{\lfloor \chi_{q_r} \times 10 \rfloor / 10}, & \text{if} \quad \xi + \rho_0 < 0 \text{ and } \chi_{q_r} \neq 0 \text{ and } \chi_{q_r} < \beta_0 \quad (0 < q_r < 1), \\ \eta_0 \lfloor \chi_{q_1} \times 10 \rfloor / 10, & \text{if} \quad \xi + \rho_0 < 0 \text{ and } \chi_{q_1} \neq 0 \text{ and } \chi_{q_1} > \beta_0 \quad (0 < q_1 < 1), \\ \eta_0, & \text{if} \quad \xi + \rho_0 > 0 \text{ or } (\xi + \gamma < 0 \text{ and } \chi_{q_1} = 0 \ (q_1 = 0)). \end{cases} \tag{5.3}$$

The values of $\eta_0 \equiv \eta$ proposed by Caeiro and Gomes (2006) for similar Fréchet and Burr models were $\eta_0 = -3$ and $\eta_0 = -1.2$, respectively (see Figures 2 to 4 of the aforementioned paper) and the correspondent $\eta_q$ values, according to (5.3), and the choices $q_1 = 0.1 < q_2 = 0.25$ are $\eta_{q_1} = \eta_{q_2} = -1.8$ for the Fréchet model with $\xi = 0.5$ and $\eta_{q_1} = \eta_{q_2} = -3$ for the Fréchet model with $\xi = 2$. For the Burr models under consideration we have chosen $\eta_{q_1} = \eta_{q_2} = -2.4$ when $(\xi, \rho_0) = (0.5, -1)$ and $\eta_{q_1} = \eta_{q_2} = -1.2$ when $(\xi, \rho_0) = (1, -0.5)$. Moreover, and due to the high volatility of the $\beta$-estimator for shifted Fréchet models, we have not represented graphically such a path.

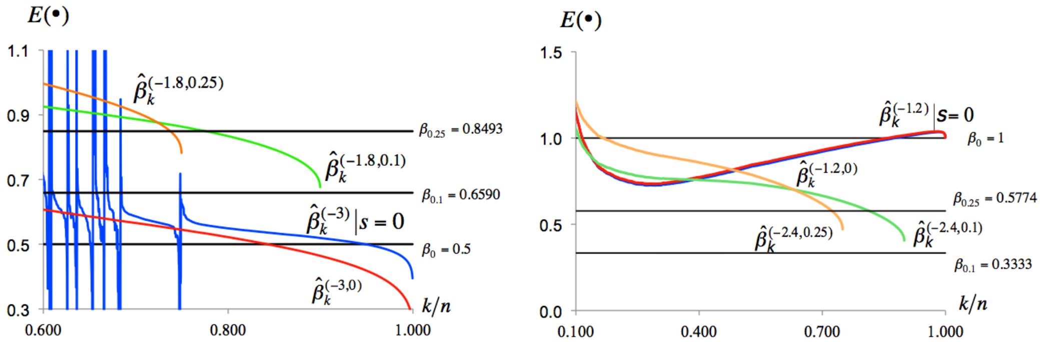


Figure 1: Mean values of the estimators under consideration for Fréchet unshifted ($s = 0$) parents, with $\xi = 0.5$ (left) and Burr unshifted ($s = 0$) parents with $(\xi, \rho_0) = (0.5, -1)$ (right), and sample size $n = 5000$.

We now would like to emphasise the following points:

- There is only a light improvement in all estimators as the sample size increases, and a high volatility of the classical $\beta$-estimators for shifted models. This is the reason why we have not represented them in the previous figures.

- For smaller sample sizes $n$, the sample paths of all estimators for small up to moderate $k$-values are even more volatile, but of the same type. Also the sample paths associated

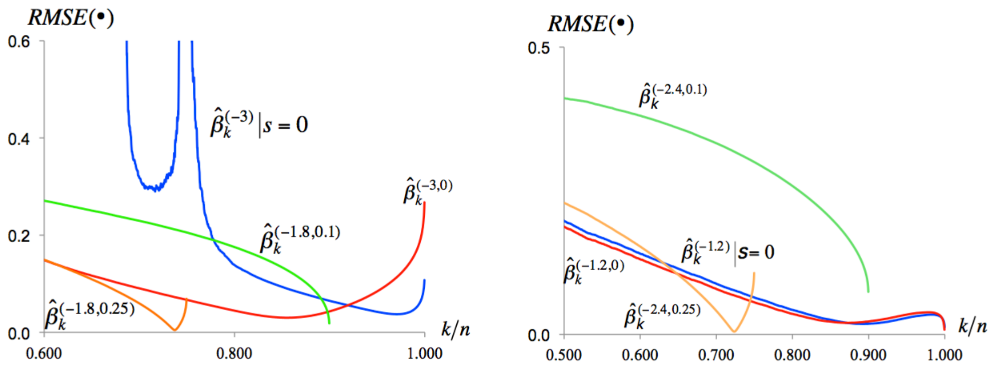


Figure 2: RMSEs of the estimators under consideration for Fréchet unshifted ($s = 0$) parents, with $\xi = 0.5$ (left) and Burr unshifted ($s = 0$) parents with $(\xi, \rho_0) = (0.5, -1)$ (right), and sample size $n = 5000$.

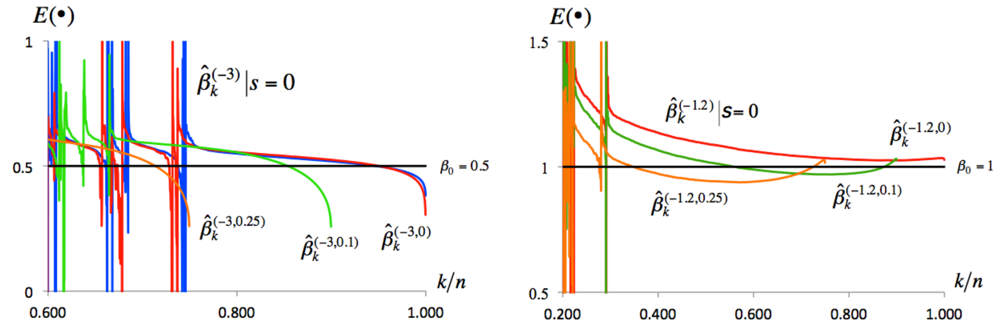


Figure 3: Mean values of the estimators under consideration for Fréchet unshifted ($s = 0$) parents, with $\xi = 2$ (left) and Burr unshifted ($s = 0$) parents with $(\xi, \rho_0) = (1, -0.5)$ (right), and sample size $n = 5000$.

with the Fréchet model are more volatile than the ones associated with the Burr model. This pattern was also detected by Caeiro and Gomes (2006) for the classical $\beta$-estimation.

- When we are in the region $\xi+\rho_0 < 0$ (see Figure 1), the PORT-$\beta$ estimator should converge to $\beta_q$, in (5.1) and (5.2), for the Fréchet and Burr models, respectively. The pattern of the PORT-$\beta$ estimators does depend on $\chi_q$, contrarily to the one of the PORT-$\rho$ estimators in Henriques-Rodrigues *et al.* (2014), making the selection a bit more intricate.

- For the Burr model the sample path of the classical $\beta$-estimator almost overlaps or even overlaps the sample path of the PORT-$\beta$ estimator associated with $q = 0$, whereas for the

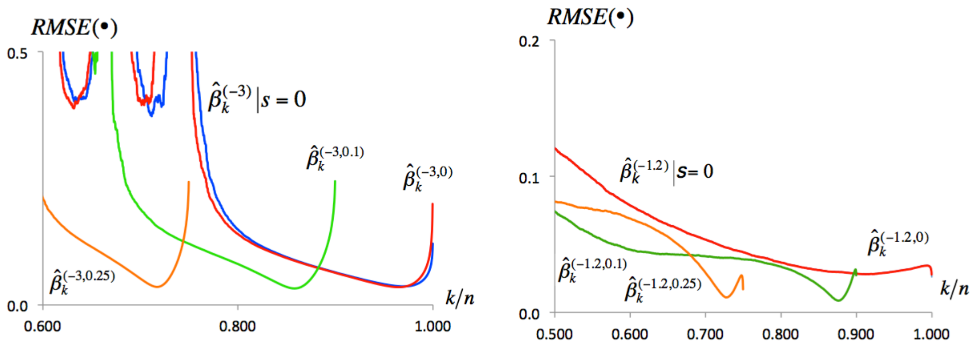


Figure 4: RMSEs of the estimators under consideration for Fréchet unshifted ($s = 0$) parents, with $\xi = 2$ (left) and Burr unshifted ($s = 0$) parents with $(\xi, \rho_0) = (1, -0.5)$ (right), and sample size $n = 5000$.

Fréchet model these same sample paths have a quite different behaviour.

- The PORT-$\beta$ estimators associated with the $\eta_q$-rule, in (5.3), are able to beat the classical ones regarding minimum RMSE, even for very large sample sizes, and when we look at moderate and large values of $k/n$ for the Fréchet model and for large values of $k/n$ for the Burr model, as can be seen in Table 1.

| Fréchet$_{0.5}$ | | | Burr$_{0.5,-1}$ | | |
|---|---|---|---|---|---|
| $\hat\beta^{\bullet}$ | $k_0^{\bullet}/n$ | $RMSE_0^{\bullet}$ | $\hat\beta^{\bullet}$ | $k_0^{\bullet}/n$ | $RMSE_0^{\bullet}$ |
| $\hat\beta^{(-3)}\|s=0$ | 0.970 | 0.0369 | $\hat\beta^{(-1.2)}\|s=0$ | 1.000 | 0.0121 |
| $\hat\beta^{(-3,0)}$ | 0.855 | 0.0297 | $\hat\beta^{(-1.2,0)}$ | 1.000 | 0.0077 |
| $\hat\beta^{(-1.8,0.1)}$ | 0.900 | 0.0181 | $\hat\beta^{(-2.4,0.1)}$ | 0.900 | 0.0736 |
| $\hat\beta^{(-1.8,0.25)}$ | 0.737 | **0.0043** | $\hat\beta^{(-2.4,0.25)}$ | 0.724 | **0.0043** |
| Fréchet$_2$ | | | Burr$_{1,-0.5}$ | | |
| $\hat\beta^{\bullet}$ | $k_0^{\bullet}/n$ | $RMSE_0^{\bullet}$ | $\hat\beta^{\bullet}$ | $k_0^{\bullet}/n$ | $RMSE_0^{\bullet}$ |
| $\hat\beta^{(-3)}\|s=0$ | 0.970 | 0.0361 | $\hat\beta^{(-1.2)}\|s=0$ | 1.000 | 0.0259 |
| $\hat\beta^{(-3,0)}$ | 0.965 | 0.0347 | $\hat\beta^{(-1.2,0)}$ | 1.000 | 0.0277 |
| $\hat\beta^{(-3,0.1)}$ | 0.858 | **0.0320** | $\hat\beta^{(-1.2,0.1)}$ | 0.875 | **0.0085** |
| $\hat\beta^{(-3,0.25)}$ | 0.718 | 0.0351 | $\hat\beta^{(-1.2,0.25)}$ | 0.727 | 0.0108 |

Table 1: Values of $k_0^{\bullet} := \mathrm{argmin}_k RMSE\big(\widehat\beta_k^{\bullet}\big)$ and $RMSE_0^{\bullet} := RMSE\big(\widehat\beta_{k_0^{\bullet}}^{\bullet}\big)$ for the different estimators and models under consideration, and a sample size $n = 5000$

- The adequate selection of the *tuning* parameters $\eta_0$ and $\eta_q$ is crucial. The choice we have used here is based on a heuristic criteria that surely is not optimal for the PORT-$\beta$ estimation, but it works for a wide variety of models. Another type of choice, similar to the one devised for the selection of $k$ through the use of the bootstrap methodology (see Gomes *et al.*, 2012, among others) is surely also an interesting topic of research, beyond the scope of this paper.

# 6 Proofs

*Proof.* [**Theorem 4.1**] (i) In the region $\mathcal{R}_1$, $A_0(t) = o(1/U_0(t))$, as $t \to \infty$, the last term of the right-hand side of (3.12) is the dominant one, and the r.v. $D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)/(1/U_0(n/k))$ converges in probability to $\alpha\eta_q\xi^{\alpha\eta_q}\chi_q\ d_{\alpha,\theta_1,\theta_2}(-\xi)$ provided that (4.1) holds. Considering the first-order approximation of the function $U(t)$, in (1.6), $U_0(n/k) \equiv C(n/k)^{\xi}$, we then get

$$\left(\frac{n}{k}\right)^{\xi} D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi) \underset{n\to\infty}{\overset{p}{\longrightarrow}} \alpha\eta_q\xi^{\alpha\eta_q}\left(\frac{\chi_q}{C}\right) d_{\alpha,\theta_1,\theta_2}(-\xi),$$

i.e. for any $r \in \mathbb{N}$

$$\left(\frac{n}{k}\right)^{r\xi} \left(D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)\right)^r \underset{n\to\infty}{\overset{p}{\longrightarrow}} \left(\alpha\eta_q\xi^{\alpha\eta_q}\left(\frac{\chi_q}{C}\right)\right)^r d_{\alpha,\theta_1,\theta_2}^r(-\xi). \tag{6.1}$$

To get rid of the unknown $\xi$ in $\left(D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)\right)^r$ it is enough to consider that

$$\left(\frac{n}{k}\right)^{\xi} D_{n,k}^{(r\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi) \underset{n\to\infty}{\overset{p}{\longrightarrow}} r\alpha\eta_q\xi^{r\alpha\eta_q}\left(\frac{\chi_q}{C}\right) d_{r\alpha,\theta_1,\theta_2}(-\xi). \tag{6.2}$$

The quotient between (6.1) and (6.2), enables us to say that

$$\left(\frac{n}{k}\right)^{\xi(r-1)} \frac{\left(D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)\right)^r}{D_{n,k}^{(r\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)} \underset{n\to\infty}{\overset{p}{\longrightarrow}} \frac{(\alpha\eta_q\,(\chi_q/C))^{r-1}}{r}\frac{(d_{\alpha,\theta_1,\theta_2})^r(-\xi)}{d_{r\alpha,\theta_1,\theta_2}(-\xi)}.$$

If we choose $r = 2$, as suggested in Caeiro and Gomes (2006), we obtain

$$\left(\frac{n}{k}\right)^{\xi} \frac{\left(D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)\right)^2}{D_{n,k}^{(2\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)} \underset{n\to\infty}{\overset{p}{\longrightarrow}} \frac{\alpha\eta_q}{2}\left(\frac{\chi_q}{C}\right)\frac{d_{\alpha,\theta_1,\theta_2}^2(-\xi)}{d_{2\alpha,\theta_1,\theta_2}(-\xi)}.$$

Since, in $\mathcal{R}_1$, $\beta_q = \chi_q/C$ and $\rho_q = -\xi$ with $(\beta_q, \rho_q)$ defined in (1.11), the class of consistent r.v.'s, that converge in probability towards $\beta_q$ for any $\alpha > 0$, $\eta_q \in \mathbb{R}$, $\theta_1, \theta_2 \neq \theta_2$, $0 \leq q < 1$ is given by

$$\widehat{\beta}_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\rho_q) := \frac{2d_{2\alpha,\theta_1,\theta_2}(\rho_q)}{\alpha\eta_q d^2_{\alpha,\theta_1,\theta_2}(\rho_q)} \left(\frac{n}{k}\right)^{-\rho_q} \frac{\left(D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)\right)^2}{D_{n,k}^{(2\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)}. \tag{6.3}$$

(ii) In the region $\xi + \rho_0 > 0$, where $1/U_0(t) = o(A_0(t))$, as $t \to \infty$, or more generally in the region $\mathcal{R}_2$, the second term of the right-hand side of (3.12) is the dominant one, i.e. $D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)/A_0(n/k)$ converges in probability to $\alpha\eta_q\xi^{\alpha\eta_q}d_{\alpha,\theta_1,\theta_2}(\rho_0)/\xi$ provided that (4.1) holds. Since we can choose $A_0(t) = \xi\beta_0 t^{\rho_0}$,

$$\left(\frac{n}{k}\right)^{-\rho_0} D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi) \underset{n\to\infty}{\overset{p}{\longrightarrow}} \beta_0\alpha\eta_q\xi^{\alpha\eta_q}d_{\alpha,\theta_1,\theta_2}(\rho_0),$$

i.e., with $r \in \mathbb{N}$

$$\left(\frac{n}{k}\right)^{-r\rho} \left(D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)\right)^r \underset{n\to\infty}{\overset{p}{\longrightarrow}} \left(\beta\alpha\eta_q\xi^{\alpha\eta_q}\right)^r \left(d_{\alpha,\theta_1,\theta_2}\right)^r(\rho). \tag{6.4}$$

Using the same type of arguments, we can get rid of the unknown $\xi$ in $\left(D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)\right)^r$ if we consider that

$$\left(\frac{n}{k}\right)^{-\rho_0} D_{n,k}^{(r\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi) \underset{n\to\infty}{\overset{p}{\longrightarrow}} \beta_0 r\alpha\eta_q\xi^{r\alpha\eta_q}d_{r\alpha,\theta_1,\theta_2}(\rho_0). \tag{6.5}$$

The quotient between (6.4) and (6.5) enables us to say that

$$\left(\frac{n}{k}\right)^{-\rho(r-1)} \frac{\left(D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)\right)^r}{D_{n,k}^{(r\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)} \underset{n\to\infty}{\overset{p}{\longrightarrow}} \frac{(\beta\alpha\eta_q)^{r-1}}{r} \frac{\left(d_{\alpha,\theta_1,\theta_2}\right)^r(\rho)}{d_{r\alpha,\theta_1,\theta_2}(\rho)}.$$

Choosing again $r = 2$, as in Caeiro and Gomes (2006), and with $\beta_q = \beta_0$ and $\rho_q = \rho_0$, $(\beta_q, \rho_q)$ given in (1.11), we get (6.3), i.e. a class of r.v.'s converging in probability to $\beta_q$ for $\alpha > 0$, $\eta_q \in \mathbb{R}$, $\theta_1 \neq \theta_2$ and $0 \leq q < 1$.

(iii) In the region $\mathcal{R}_3$, $A_0(t)$ and $1/U_0(t)$ are of the same order, i.e. the dominant terms of the right-hand side of (3.12) are the second and the last. If we assume that (4.1) holds,

$$\frac{D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)}{A_0(n/k)} \underset{n\to\infty}{\overset{p}{\longrightarrow}} \frac{\alpha\eta_q\xi^{\alpha\eta_q}}{\xi}\left[d_{\alpha,\theta_1,\theta_2}(\rho_0) + \frac{\chi_q}{\beta_0 C}d_{\alpha,\theta_1,\theta_2}(-\xi)\right].$$

Since $A_0(t) = \xi\beta_0 t^{\rho_0}$ we then get

$$\left(\frac{n}{k}\right)^{-\rho_0} D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi) \underset{n\to\infty}{\overset{p}{\longrightarrow}} \alpha\beta_0\eta_q\xi^{\alpha\eta_q}\left(d_{\alpha,\theta_1,\theta_2}(\rho_0) + \frac{\chi_q}{\beta_0 C} d_{\alpha,\theta_1,\theta_2}(-\xi)\right).$$

But in $\mathcal{R}_3$, $\rho_0 = -\xi$, thence

$$\left(\frac{n}{k}\right)^{-\rho_0} D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi) \underset{n\to\infty}{\overset{p}{\longrightarrow}} \alpha\eta_q\xi^{\alpha\eta_q}\left(\beta_0 + \frac{\chi_q}{C}\right) d_{\alpha,\theta_1,\theta_2}(\rho_0).$$

Considering the same type of procedures used in cases (i) and (ii) and with $r = 2$ we are led to

$$\left(\frac{n}{k}\right)^{-\rho_0} \frac{\left(D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)\right)^2}{D_{n,k}^{(2\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)} \underset{n\to\infty}{\overset{p}{\longrightarrow}} \frac{\alpha\eta_q}{2}\left(\beta_0 + \frac{\chi_q}{C}\right)\frac{d_{\alpha,\theta_1,\theta_2}^2(\rho_0)}{d_{2\alpha,\theta_1,\theta_2}(\rho_0)},$$

and with $\beta_q = \beta_0 + \chi_q/C$, from (1.11), we get (6.3) and consistency follows.
The results presented in cases (i), (ii) and (iii) still hold true if we replace $\rho_q$ by any consistent estimator of $\rho_q$, $\widehat{\rho}^{(q)}$, such that $(\widehat{\rho}^{(q)} - \rho_q)\ln(n/k) = o_p(1)$. □

*Proof.* [**Theorem 4.2**]. (i) In the region $\mathcal{R}_1$, $A_0(t) = o(1/U_0(t))$, as $t \to \infty$, the third and last term of the right-hand side of (3.12) is the dominant one, and the r.v.'s $D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}(\xi)/(1/U_0(n/k))$ converge in probability to $\alpha\ \eta_q\xi^{\alpha\eta_q}\chi_q\ d_{\alpha,\theta_1,\theta_2}(-\xi)$ provided that (4.1) holds, i.e. if $\sqrt{k}/U_0(n/k) \to \infty$, as $n \to \infty$ (see Remark 4.1). Moreover, we get

$$\begin{aligned}\frac{D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)}{1/U_0(n/k)} &\overset{d}{=} \alpha\ \eta_q\xi^{\alpha\eta_q}\chi_q\ d_{\alpha,\theta_1,\theta_2}(-\xi)\Bigg(1 + \frac{1}{\alpha\chi_q d_{\alpha,\theta_1,\theta_2}(-\xi)\sqrt{k}}\ W_k^{(\alpha,\theta_1,\theta_2)}U_0(n/k) \\ &+ \frac{d_{\alpha,\theta_1,\theta_2}(\rho_0)}{\xi\chi_q d_{\alpha,\theta_1,\theta_2}(-\xi)}A_0(n/k)U_0(n/k)(1+o_p(1)) + \frac{\chi_q\ y_{\alpha,\theta_1,\theta_2,\eta_q}(\xi)}{d_{\alpha,\theta_1,\theta_2}(-\xi)U_0(n/k)}(1+o_p(1))\Bigg).\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{\left(D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)\right)^2}{1/U_0^2(n/k)} &\overset{d}{=} (\alpha\ \eta_q)^2\,\xi^{2\alpha\eta_q}\chi_q^2\ d_{\alpha,\theta_1,\theta_2}^2(-\xi)\Bigg(1 + \frac{2}{\alpha\chi_q d_{\alpha,\theta_1,\theta_2}(-\xi)\sqrt{k}}\ W_k^{(\alpha,\theta_1,\theta_2)}U_0(n/k) \\ &+ 2\frac{d_{\alpha,\theta_1,\theta_2}(\rho_0)}{\xi\chi_q d_{\alpha,\theta_1,\theta_2}(-\xi)}A_0(n/k)U_0(n/k)(1+o_p(1)) + \frac{2\chi_q\ y_{\alpha,\theta_1,\theta_2,\eta_q}(\xi)}{d_{\alpha,\theta_1,\theta_2}(-\xi)U_0(n/k)}(1+o_p(1))\Bigg),\end{aligned} \tag{6.6}$$

and since $1/(1+x) = 1 - x + o(x)$, as $x \to 0$, we get

$$\begin{aligned}\frac{1/U_0(n/k)}{D_{n,k}^{(2\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)} &\overset{d}{=} \frac{1}{2\alpha\ \eta_q\xi^{2\alpha\eta_q}\chi_q\ d_{2\alpha,\theta_1,\theta_2}(-\xi)}\Bigg(1 - \frac{1}{2\alpha\chi_q d_{2\alpha,\theta_1,\theta_2}(-\xi)\sqrt{k}}\ W_k^{(2\alpha,\theta_1,\theta_2)}U_0(n/k) \\ &- \frac{d_{2\alpha,\theta_1,\theta_2}(\rho_0)}{\xi\chi_q d_{2\alpha,\theta_1,\theta_2}(-\xi)}A_0(n/k)U_0(n/k)(1+o_p(1)) - \frac{\chi_q\ y_{2\alpha,\theta_1,\theta_2,\eta_q}(\xi)}{d_{2\alpha,\theta_1,\theta_2}(-\xi)U_0(n/k)}(1+o_p(1))\Bigg).\end{aligned} \tag{6.7}$$

The quotient between (6.6) and (6.7) enables us to say that

$$\frac{\left(D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)\right)^2}{D_{n,k}^{(2\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)/U_0(n/k)} \overset{d}{=} \frac{\alpha\ \eta_q\chi_q}{2}\frac{d^2_{\alpha,\theta_1,\theta_2}(-\xi)}{d_{2\alpha,\theta_1,\theta_2}(-\xi)}\Bigg(1+\frac{2}{\alpha\chi_q d_{\alpha,\theta_1,\theta_2}(-\xi)\sqrt{k}}\ W_k^{(\alpha,\theta_1,\theta_2)}U_0(n/k)$$

$$+2\frac{d_{\alpha,\theta_1,\theta_2}(\rho_0)}{\xi\chi_q d_{\alpha,\theta_1,\theta_2}(-\xi)}A_0(n/k)U_0(n/k)(1+o_p(1))+\frac{2\chi_q\ y_{\alpha,\theta_1,\theta_2,\eta_q}(\xi)}{d_{\alpha,\theta_1,\theta_2}(-\xi)U_0(n/k)}(1+o_p(1))\Bigg)$$

$$\times\Bigg(1-\frac{1}{2\alpha\chi_q d_{2\alpha,\theta_1,\theta_2}(-\xi)\sqrt{k}}\ W_k^{(2\alpha,\theta_1,\theta_2)}U_0(n/k)$$

$$-\frac{d_{2\alpha,\theta_1,\theta_2}(\rho_0)}{\xi\chi_q d_{2\alpha,\theta_1,\theta_2}(-\xi)}A_0(n/k)U_0(n/k)(1+o_p(1))-\frac{\chi_q\ y_{2\alpha,\theta_1,\theta_2,\eta_q}(\xi)}{d_{2\alpha,\theta_1,\theta_2}(-\xi)U_0(n/k)}(1+o_p(1))\Bigg).$$

Since, in $\mathcal{R}_1$, $\beta_q=\chi_q/C$, $\rho_q=-\xi$ with $(\beta_q,\rho_q)$ defined in (1.11), and we can write

$$\widehat{\beta}_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\rho_q):=\frac{2d_{2\alpha,\theta_1,\theta_2}(\rho_q)}{\alpha\eta_q d^2_{\alpha,\theta_1,\theta_2}(\rho_q)}\left(\frac{n}{k}\right)^{-\rho_q}\frac{\left(D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)\right)^2}{D_{n,k}^{(2\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)}$$

$$=\beta_q\Bigg(1+\frac{2\ W_k^{(\alpha,\theta_1,\theta_2)}U_0(n/k)}{\alpha\chi_q d_{\alpha,\theta_1,\theta_2}(-\xi)\sqrt{k}}-\frac{W_k^{(2\alpha,\theta_1,\theta_2)}U_0(n/k)}{2\alpha\chi_q d_{2\alpha,\theta_1,\theta_2}(-\xi)\sqrt{k}}$$

$$+\frac{\chi_q y^{(\alpha,\theta_1,\theta_2,\eta_q)}(\xi,-\xi)}{U_0(n/k)}(1+o_p(1))+\frac{z^{(\alpha,\theta_1,\theta_2)}(\rho_0,-\xi)}{\xi\chi_q}A_0(n/k)U_0(n/k)(1+o_p(1))\Bigg),$$

with $y^{(\alpha,\theta_1,\theta_2,\eta_q)}(\xi,-\xi)$ and $z^{(\alpha,\theta_1,\theta_2)}(\rho,-\xi)$ defined in (4.4) and (4.5), respectively.

If we consider sequences of positive intermediate integers $k=k_n$ such that $k_n=o(n)$, $\sqrt{k}/U_0(n/k)\to\infty$, $\sqrt{k}A_0(n/k)\to\lambda$ and $\sqrt{k}/U_0^2(n/k)\to\lambda_{_U}$, as $n\to\infty$, and noticing that with $\chi_q\neq 0$, given in (1.9), $A_0(t)U_0(t)=o(1/U_0(t))$, in $\mathcal{R}_{11}$, $1/U_0(t)=o(A_0(t)U_0(t))$, in $\mathcal{R}_{13}$ and $A_0(t)U_0(t)=O(1/U_0(t))$, in $\mathcal{R}_{12}$, the result holds.

(ii) In the region $\xi+\rho_0>0$, where $1/U_0(t)=o(A_0(t))$, as $t\to\infty$, or more generally in the region $\mathcal{R}_2$, the second term of the right-hand side of (3.12) is the dominant one. In $\mathcal{R}_2$, $A_q(t)=A_0(t)$, so condition (4.1) can be rewritten as $\sqrt{k}A_0(n/k)\to\infty$, as $n\to\infty$ and if we assume that this condition holds,

$$\frac{D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)}{A_0(n/k)} \overset{d}{=} \alpha\eta_q\xi^{\alpha\eta_q-1}d_{\alpha,\theta_1,\theta_2}(\rho_0)\Bigg(1+\frac{\xi W_k^{(\alpha,\theta_1,\theta_2)}}{\alpha d_{\alpha,\theta_1,\theta_2}(\rho_0)\sqrt{k}A(n/k)}$$

$$+\frac{u_{\alpha,\theta_1,\theta_2,\eta_q}(\rho_0)}{d_{\alpha,\theta_1,\theta_2}(\rho_0)}A_0(n/k)(1+o_p(1))+\frac{v_{\alpha,\theta_1,\theta_2}(\rho_0,\rho_0')}{d_{\alpha,\theta_1,\theta_2}(\rho_0)}B_0(n/k)(1+o_p(1))$$

$$+\frac{\xi\chi_q d_{\alpha,\theta_1,\theta_2}(-\xi)}{d_{\alpha,\theta_1,\theta_2}(\rho_0)A_0(n/k)U_0(n/k)}(1+o_p(1))\Bigg).$$

Therefore,

$$\frac{\left(D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)\right)^2}{A_0^2(n/k)} \stackrel{d}{=} (\alpha\eta_q)^2\xi^{2\alpha\eta_q-2}d^2_{\alpha,\theta_1,\theta_2}(\rho_0)\Bigg(1+\frac{2\xi W_k^{(\alpha,\theta_1,\theta_2)}}{\alpha d_{\alpha,\theta_1,\theta_2}(\rho_0)\sqrt{k}A(n/k)}$$
$$+\frac{2u_{\alpha,\theta_1,\theta_2,\eta_q}(\rho_0)}{d_{\alpha,\theta_1,\theta_2}(\rho_0)}A_0(n/k)(1+o_p(1))+\frac{2v_{\alpha,\theta_1,\theta_2}(\rho_0,\rho_0')}{d_{\alpha,\theta_1,\theta_2}(\rho_0)}B_0(n/k)(1+o_p(1))$$
$$+\frac{2\xi\chi_q d_{\alpha,\theta_1,\theta_2}(-\xi)}{d_{\alpha,\theta_1,\theta_2}(\rho_0)A_0(n/k)U_0(n/k)}(1+o_p(1))\Bigg), \quad (6.8)$$

and given that $1/(1+x)=1-x+o(x)$, as $x\to 0$, we obtain

$$\frac{A_0(n/k)}{D_{n,k}^{(2\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)} \stackrel{d}{=} \frac{1}{2\alpha\eta_q\xi^{2\alpha\eta_q-1}d_{2\alpha,\theta_1,\theta_2}(\rho_0)}\Bigg(1-\frac{\xi W_k^{(2\alpha,\theta_1,\theta_2)}}{2\alpha d_{2\alpha,\theta_1,\theta_2}(\rho_0)\sqrt{k}A(n/k)}$$
$$-\frac{u_{2\alpha,\theta_1,\theta_2,\eta_q}(\rho_0)}{d_{2\alpha,\theta_1,\theta_2}(\rho_0)}A_0(n/k)(1+o_p(1))-\frac{v_{2\alpha,\theta_1,\theta_2}(\rho_0,\rho_0')}{d_{2\alpha,\theta_1,\theta_2}(\rho_0)}B_0(n/k)(1+o_p(1))$$
$$-\frac{\xi\chi_q d_{2\alpha,\theta_1,\theta_2}(-\xi)}{d_{2\alpha,\theta_1,\theta_2}(\rho_0)A_0(n/k)U_0(n/k)}(1+o_p(1))\Bigg). \quad (6.9)$$

The quotient between (6.8) and (6.9) enables us to say that

$$\frac{\left(D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)\right)^2}{D_{n,k}^{(2\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)A_0(n/k)} \stackrel{d}{=} \frac{\alpha\eta_q}{2\xi}\frac{d^2_{\alpha,\theta_1,\theta_2}(\rho_0)}{d_{2\alpha,\theta_1,\theta_2}(\rho_0)}\Bigg(1+\frac{2\xi W_k^{(\alpha,\theta_1,\theta_2)}}{\alpha d_{\alpha,\theta_1,\theta_2}(\rho_0)\sqrt{k}A(n/k)}$$
$$+\frac{2u_{\alpha,\theta_1,\theta_2,\eta_q}(\rho_0)}{d_{\alpha,\theta_1,\theta_2}(\rho_0)}A_0(n/k)(1+o_p(1))+\frac{2v_{\alpha,\theta_1,\theta_2}(\rho_0,\rho_0')}{d_{\alpha,\theta_1,\theta_2}(\rho_0)}B_0(n/k)(1+o_p(1))$$
$$+\frac{2\xi\chi_q d_{\alpha,\theta_1,\theta_2}(-\xi)}{d_{\alpha,\theta_1,\theta_2}(\rho_0)A_0(n/k)U_0(n/k)}(1+o_p(1))\Bigg)\times\Bigg(1-\frac{\xi W_k^{(2\alpha,\theta_1,\theta_2)}}{2\alpha d_{2\alpha,\theta_1,\theta_2}(\rho_0)\sqrt{k}A(n/k)}$$
$$-\frac{u_{2\alpha,\theta_1,\theta_2,\eta_q}(\rho_0)}{d_{2\alpha,\theta_1,\theta_2}(\rho_0)}A_0(n/k)(1+o_p(1))-\frac{v_{2\alpha,\theta_1,\theta_2}(\rho_0,\rho_0')}{d_{2\alpha,\theta_1,\theta_2}(\rho_0)}B_0(n/k)(1+o_p(1))$$
$$-\frac{\xi\chi_q d_{2\alpha,\theta_1,\theta_2}(-\xi)}{d_{2\alpha,\theta_1,\theta_2}(\rho_0)A_0(n/k)U_0(n/k)}(1+o_p(1))\Bigg).$$

But, in $\mathcal{R}_2$, $\beta_q=\beta_0$, $\rho_q=\rho_0$ with $(\beta_q,\rho_q)$ defined in (1.11), and we get

$$\widehat{\beta}_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\rho_q) := \frac{2d_{2\alpha,\theta_1,\theta_2}(\rho_q)}{\alpha\eta_q d^2_{\alpha,\theta_1,\theta_2}(\rho_q)}\left(\frac{n}{k}\right)^{-\rho_q}\frac{\left(D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)\right)^2}{D_{n,k}^{(2\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)}$$
$$=\beta_q\Bigg(1+\frac{2\xi W_k^{(\alpha,\theta_1,\theta_2)}}{\alpha d_{\alpha,\theta_1,\theta_2}(\rho_0)\sqrt{k}A_0(n/k)}-\frac{\xi W_k^{(2\alpha,\theta_1,\theta_2)}}{2\alpha d_{2\alpha,\theta_1,\theta_2}(\rho_0)\sqrt{k}A_0(n/k)}$$
$$+u^{(\alpha,\theta_1,\theta_2,\eta_q)}(\rho_0)A_0(n/k)(1+o_p(1))+v^{(\alpha,\theta_1,\theta_2)}(\rho_0,\rho_0')B_0(n/k)(1+o_p(1))$$
$$+\frac{\xi\chi_q z^{(\alpha,\theta_1,\theta_2)}(-\xi,\rho_0)}{A_0(n/k)U_0(n/k)}(1+o_p(1))\Bigg),$$

with $z^{(\alpha,\theta_1,\theta_2)}(\rho,-\xi)$, $u^{(\alpha,\theta_1,\theta_2,\eta_q)}(\rho)$ and $v^{(\alpha,\theta_1,\theta_2)}(\rho,\rho')$ defined in (4.5), (4.6) and (4.7), respectively.

Considering sequences of positive intermediate integers $k = k_n$ such that $k_n = o(n)$, $\sqrt{k}/U_0(n/k) \to \infty$, and if we further assume that $\sqrt{k}A_0^2(n/k) \to \lambda_A$, $\sqrt{k}A_0(n/k)B_0(n/k) \to \lambda_B$ and $\sqrt{k}/U_0(n/k) \to \lambda'$, as $n \to \infty$, then, with $\chi_q \neq 0$ given in (1.9) and $A_0(t) = O(B_0(t))$, $A_0(t) = o(1/(A_0(t)U_0(t)))$, in $\mathcal{R}_{21}$, $1/(A_0(t)U_0(t)) = o(A_0(t))$, in $\mathcal{R}_{23}$ and $A_0(t) = O(1/(A_0(t)U_0(t))$, in $\mathcal{R}_{22}$, the result follows.

(iii) In the region $\mathcal{R}_3$, $A_0(t)$ and $1/U_0(t)$ are of the same order, i.e. the dominant terms of the right-hand side of (3.12) are the second and the third. In $\mathcal{R}_3$, $A_q(t) = A_0(t) + \xi\chi_q/U_0(t)$, so condition (4.1) can be rewritten as $\sqrt{k}A_0(n/k) \to \infty$, as $n \to \infty$ or $\sqrt{k}/U_0(n/k) \to \infty$, as $n \to \infty$. If we assume that the first condition holds with $\lim_{n\to\infty} 1/(A_0(n/k)U_0(n/k)) := \widetilde{\lambda} = \frac{1}{\xi\beta_0 C}$, then

$$\begin{aligned}
\frac{D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)}{A_0(n/k)} &\overset{d}{=} \alpha\eta_q\xi^{\alpha\eta_q-1}\left(1+\xi\widetilde{\lambda}\chi_q\right)d_{\alpha,\theta_1,\theta_2}(\rho_0)\Bigg(1+\frac{\xi W_k^{(\alpha,\theta_1,\theta_2)}}{\alpha d_{\alpha,\theta_1,\theta_2}(\rho_0)\big(1+\xi\widetilde{\lambda}\chi_q\big)\sqrt{k}A(n/k)} \\
&\quad+\frac{u_{\alpha,\theta_1,\theta_2,\eta_q}(\rho_0)}{(1+\xi\widetilde{\lambda}\chi_q)d_{\alpha,\theta_1,\theta_2}(\rho_0)}A_0(n/k)(1+o_p(1))+\frac{v_{\alpha,\theta_1,\theta_2}(\rho_0,\rho_0')}{(1+\xi\widetilde{\lambda}\chi_q)d_{\alpha,\theta_1,\theta_2}(\rho_0)}B_0(n/k)(1+o_p(1)) \\
&\quad+\frac{\xi\chi_q\ w_{\alpha,\theta_1,\theta_2,\eta_q}(\xi,\rho_0)}{\big(1+\xi\widetilde{\lambda}\chi_q\big)d_{\alpha,\theta_1,\theta_2}(\rho_0)U_0(n/k)}(1+o_p(1))+\frac{\xi\chi_q^2 y_{\alpha,\theta_1,\theta_2,\eta_q}(\xi)}{\big(1+\xi\widetilde{\lambda}\chi_q\big)d_{\alpha,\theta_1,\theta_2}(\rho_0)A_0(n/k)U_0^2(n/k)}\Bigg).
\end{aligned}$$

Considering the same type of procedures used in cases (i) and (ii) we are led to

$$\begin{aligned}
\frac{\left(D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)\right)^2}{D_{n,k}^{(2\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)A_0(n/k)} &\overset{d}{=} \frac{\alpha\eta_q}{2\xi}\left(1+\xi\widetilde{\lambda}\chi_q\right)\frac{d^2_{\alpha,\theta_1,\theta_2}(\rho_0)}{d_{2\alpha,\theta_1,\theta_2}(\rho_0)}\Bigg(1+\frac{2\xi W_k^{(\alpha,\theta_1,\theta_2)}}{\alpha d_{\alpha,\theta_1,\theta_2}(\rho_0)\big(1\xi\widetilde{\lambda}\chi_q\big)\sqrt{k}A(n/k)} \\
&\quad+\frac{2u_{\alpha,\theta_1,\theta_2,\eta_q}(\rho_0)}{(1+\xi\widetilde{\lambda}\chi_q)d_{\alpha,\theta_1,\theta_2}(\rho_0)}A_0(n/k)(1+o_p(1))+\frac{2v_{\alpha,\theta_1,\theta_2}(\rho_0,\rho_0')}{(1+\xi\widetilde{\lambda}\chi_q)d_{\alpha,\theta_1,\theta_2}(\rho_0)}B_0(n/k)(1+o_p(1)) \\
&\quad+\frac{2\xi\chi_q\ w_{\alpha,\theta_1,\theta_2,\eta_q}(\xi,\rho_0)}{\big(1+\xi\widetilde{\lambda}\chi_q\big)d_{\alpha,\theta_1,\theta_2}(\rho_0)U_0(n/k)}(1+o_p(1))+\frac{2\xi\chi_q^2 y_{\alpha,\theta_1,\theta_2,\eta_q}(\xi)}{\big(1+\xi\widetilde{\lambda}\chi_q\big)d_{\alpha,\theta_1,\theta_2}(\rho_0)A_0(n/k)U_0^2(n/k)}\Bigg) \\
&\quad\times\Bigg(1-\frac{\xi W_k^{(2\alpha,\theta_1,\theta_2)}}{2\alpha d_{2\alpha,\theta_1,\theta_2}(\rho_0)\big(1+\xi\widetilde{\lambda}\chi_q\big)\sqrt{k}A(n/k)} \\
&\quad-\frac{u_{2\alpha,\theta_1,\theta_2,\eta_q}(\rho_0)}{(1+\xi\widetilde{\lambda}\chi_q)d_{2\alpha,\theta_1,\theta_2}(\rho_0)}A_0(n/k)(1+o_p(1))-\frac{v_{2\alpha,\theta_1,\theta_2}(\rho_0,\rho_0')}{(1+\xi\widetilde{\lambda}\chi_q)d_{2\alpha,\theta_1,\theta_2}(\rho_0)}B_0(n/k)(1+o_p(1)) \\
&\quad-\frac{\xi\chi_q\ w_{2\alpha,\theta_1,\theta_2,\eta_q}(\xi,\rho_0)}{\big(1+\xi\widetilde{\lambda}\chi_q\big)d_{2\alpha,\theta_1,\theta_2}(\rho_0)U_0(n/k)}(1+o_p(1))-\frac{\xi\chi_q^2 y_{2\alpha,\theta_1,\theta_2,\eta_q}(\xi)}{\big(1+\xi\widetilde{\lambda}\chi_q\big)d_{2\alpha,\theta_1,\theta_2}(\rho_0)A_0(n/k)U_0^2(n/k)}\Bigg).
\end{aligned}$$

In $\mathcal{R}_3$, we have $\beta_q = \beta_0 + \frac{\chi_q}{C}$ and $\rho_q = \rho_0$ with $(\beta_q, \rho_q)$ defined in (1.11), therefore,

$$\begin{aligned}\widehat{\beta}_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\rho_q) &:= \frac{2d_{2\alpha,\theta_1,\theta_2}(\rho_q)}{\alpha\eta_q d^2_{\alpha,\theta_1,\theta_2}(\rho_q)}\left(\frac{n}{k}\right)^{-\rho_q}\frac{\left(D_{n,k}^{(\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)\right)^2}{D_{n,k}^{(2\alpha,\theta_1,\theta_2,\eta_q,q)}(\xi)}\\
&= \beta_q\bigg(1 + \frac{2\xi W_k^{(\alpha,\theta_1,\theta_2)}}{\alpha\big(1+\xi\widetilde{\lambda}\chi_q\big)d_{\alpha,\theta_1,\theta_2}(\rho_0)\sqrt{k}A_0(n/k)} - \frac{\xi W_k^{(2\alpha,\theta_1,\theta_2)}}{2\alpha\big(1+\xi\widetilde{\lambda}\chi_q\big)d_{2\alpha,\theta_1,\theta_2}(\rho_0)\sqrt{k}A_0(n/k)}\\
&\quad + \frac{u^{(\alpha,\theta_1,\theta_2,\eta_q)}(\rho_0)}{1+\xi\widetilde{\lambda}\chi_q}A_0(n/k)(1+o_p(1)) + \frac{v^{(\alpha,\theta_1,\theta_2)}(\rho_0,\rho_0')}{1+\xi\widetilde{\lambda}\chi_q}B_0(n/k)(1+o_p(1))\\
&\qquad + \frac{\xi\chi_q w^{(\alpha,\theta_1,\theta_2,\eta_q)}(\xi,\rho_0)}{\big(1+\xi\widetilde{\lambda}\chi_q\big)U_0(n/k)}(1+o_p(1)) + \frac{\xi\chi_q^2 y^{(\alpha,\theta_1,\theta_2,\eta_q)}(\xi,\rho_0)}{\big(1+\xi\widetilde{\lambda}\chi_q\big)A_0(n/k)U_0^2(n/k)}(1+o_p(1))\bigg),\end{aligned}$$

with $y^{(\alpha,\theta_1,\theta_2,\eta_q)}(\xi,\rho)$, $z^{(\alpha,\theta_1,\theta_2)}(\rho,-\xi)$, $u^{(\alpha,\theta_1,\theta_2,\eta_q)}(\rho)$ and $v^{(\alpha,\theta_1,\theta_2)}(\rho,\rho')$ defined in (4.4), (4.5), (4.6) and (4.7), respectively.

The proof of the theorem follows for sequences of positive intermediate integers $k = k_n$ such that $k_n = o(n)$, $\sqrt{k}A_0(n/k) \to \infty$, $\sqrt{k}A_0^2(n/k) \to \lambda_{_A}$, $\sqrt{k}A_0(n/k)B_0(n/k) \to \lambda_{_B}$, $\sqrt{k}A_0(n/k)/U_0(n/k) \to \lambda_{AU}$ and $\sqrt{k}/U_0^2(n/k) \to \lambda_U$, as $n \to \infty$, and taking into account that $\beta_q/(1+\xi\widetilde{\lambda}\chi_q) = \beta_0$.

The results presented in cases (i), (ii) and (iii) still hold true if we replace $\rho_q$ by any consistent estimator of $\rho_q$, $\widehat{\rho}^{(q)}$, such that $(\widehat{\rho}^{(q)} - \rho_q)\ln(n/k) = o_p(1)$. □

**Remark 6.1.** *The replacement of* $\rho$ *by* $\widehat{\rho}^{(q)}$ *in the scale factor* $2d_{2\alpha,\theta_1,\theta_2}(\rho_q)/\big(\alpha\eta_q d^2_{\alpha,\theta_1,\theta_2}(\rho)\big)$ *places no problem due to a continuity argument, provided that* $\widehat{\rho}^{(q)}$ *is consistent for the estimation of* $\rho_q$*. However, the replacement of* $\rho_q$ *by* $\widehat{\rho}^{(q)}$ *in* $(k/n)^{\rho_q}$ *requires that* $(k/n)^{\rho_q}/(k/n)^{\widehat{\rho}^{(q)}} \underset{n\to\infty}{\overset{p}{\longrightarrow}} 1$ *and hence the need to impose the condition* $\big(\widehat{\rho}^{(q)} - \rho_q\big)\ln(n/k) = o_p(1)$*.*

# Appendix: The second and third-order frameworks for heavy-tailed models under a non-null shift

As mentioned above, if we induce any arbitrary shift, $s \in \mathbb{R}\backslash\{0\}$, in the unshifted model $X_0$, with quantile function $U_0(t)$, the transformed r.v., $X_s = X_0 + s$, has an associated quantile function given by $U_s(t) = U_0(t) + s$. The second and third-order conditions in (1.4) and (1.5), respectively, can then be rewritten as

$$\lim_{t\to\infty} \frac{\ln U_s(tx) - \ln U_s(t) - \xi \ln x}{A_s(t)} = \psi_{\rho_s}(x) := \begin{cases} \frac{x^{\rho_s}-1}{\rho_s}, & \text{if} \quad \rho_s < 0, \\ \ln x, & \text{if} \quad \rho_s = 0, \end{cases} \tag{6.10}$$

and

$$\lim_{t\to\infty} \frac{\frac{\ln U_s(tx)-\ln U_s(t)-\xi \ln x}{A_s(t)} - \psi_{\rho_s}(x)}{B_s(t)} = \begin{cases} \frac{x^{\rho_s+\rho'_s}-1}{\rho_s+\rho'_s}, & \text{if} \quad \min(\rho_s, \rho'_s) < 0, \\ \ln x, & \text{if} \quad \rho_s = \rho'_s = 0, \end{cases} \tag{6.11}$$

and hold for all $x > 0$, with $|A_s| \in RV_{\rho_s}$, $|B_s| \in RV_{\rho'_s}$, $\rho_s,\ \rho'_s < 0$. As a replacement of Lemma 3.1, if we assume that $U_0 \in RV_\xi$ satisfies the second-order condition (1.4) with $\rho = \rho_0$ and $A = A_0$. Then $U_s(t) := U_0(t) + s$ is such that $U_s \in RV_\xi$ and (6.10) holds with

$$A_s(t) := \begin{cases} -\xi s/U_0(t), & \text{if} \quad \xi + \rho_0 < 0 \text{ and } s \neq 0, \\ A_0(t) - \xi s/U_0(t), & \text{if} \quad \xi + \rho_0 = 0 \text{ and } s \neq 0, \\ A_0(t), & \text{if} \quad \xi + \rho_0 > 0 \text{ or } s = 0, \end{cases} \tag{6.12}$$

and $(\beta_s, \rho_s)$ given in (1.7).

Consequently, the introduction of a shift in the model underlying the data can possibly change the shape second-order parameter $\rho = \rho_s$, in (1.4), which is indeed equal to $-\xi$ whenever we induce a non-null shift in any unshifted model with $\xi + \rho_0 < 0$, as, for instance, $X \equiv X_0 \frown$ Fréchet($\xi = 0.25$), for which $\rho_0 = -1$. Then, and for $X_s = X_0 + s$, $s \neq 0$, the second-order parameter $\rho$, in (1.4), becomes $-\xi$. In the sequel, and for a reasonably large set $\mathcal{H}$ of heavy-tailed models, $\mathcal{H} \subset \mathcal{D}_\mathcal{M}(G_{\xi>0})$, we shall analyse the impact of a shift $s \neq 0$ not only on $(\beta, \rho)$ and $A(\cdot)$, but, more generally, in the vector of unknown parameters $(\beta, \rho, \beta', \rho')$, with $(\beta', \rho')$ the scale and shape third-order parameters, proceeding to a characterisation of $(\beta_s, \rho_s, \beta'_s, \rho'_s)$ and the functionals $U_s(t)$, $A_s(t)$ and $B_s(t)$, comparatively with the functionals $U_0(t)$, $A_0(t)$ and $B_0(t)$ corresponding to an unshifted model.

### A subclasse of Hall-Welsh class of models

The so-called Hall-Welsh class of models was first introduced in Hall (1982), later used in Hall and Welsh (1985) with a restriction $E_1 \neq 0$, and it is now used under a third-order framework. We thus assume to be working in a class $\mathcal{H}$ of heavy-tailed models, such that

$$\overline{F}(x) \equiv \overline{F}_0(x) = (x/C)^{-1/\xi} \Big\{1 + E_1\,(x/C)^{\rho_0/\xi} + E_2\,(x/C)^{2\rho_0/\xi} + o\big(x^{2\rho_0/\xi}\big)\Big\},$$

as $x \to \infty$, where $\xi > 0$, $\rho_0 < 0$, $C > 0$ and $E_1, E_2 \neq 0$. Equivalently, we can say that, as $t \to \infty$,

$$U(t) \equiv U_0(t) = Ct^{\xi}\left\{1 + D_1 t^{\rho_0} + D_2 t^{2\rho_0} + o(t^{2\rho_0})\right\}, \tag{6.13}$$

where $D_1 = \xi E_1$ and $D_2 = \xi\left(E_2 + (2\rho_0 + \xi - 1)/2E_1^2\right)$. Then, the third-order condition in (1.5) holds, with $\rho = \rho' = \rho_0$, $A_0(t) = \rho_0 D_1 t^{\rho_0}$ and $B_0(t) = (2D_2/D_1 - D_1)\, t^{\rho_0}$. For this class of models and choosing the parameterizations $A_0(t) := \xi\beta_0 t^{\rho_0}$ and $B_0(t) := \beta_0' t^{\rho_0}$, we have $D_1 = \xi\beta_0/\rho_0$ and $D_2 = D_1(D_1 + \beta_0')/2$.

**Remark 6.2.** *The* log-gamma *model ($\rho = 0$ in* (1.4)*) is out of the class of models in* (6.13)*. The unit* Pareto *model, with d.f. $F(z) = 1 - z^{-1/\xi}$, $z \geq 1$, and quantile function $U(t) = t^{\xi}$, $t \geq 1$, is also out of this class of models. Indeed, we get $U(tx)/U(t) = x^{\xi}$ for all $x \geq 1$, i.e. $A(t) \equiv 0$ in* (1.4) *meaning that we may assume the fastest convergence attached to $\rho = -\infty$.*

If we induce a deterministic shift, $s \in \mathbb{R}\backslash\{0\}$, in the underlying model, the associated reciprocal quantile function, $U_s(t)$, is then given by,

$$U_s(t) = Ct^{\xi}\left\{1 + D_1 t^{\rho_0} + D_2 t^{2\rho_0} + sC^{-1}t^{-\xi} + o(t^{2\rho_0})\right\}, \quad \text{as} \quad t \to \infty.$$

The parameter $(\beta = \beta_s, \rho = \rho_s)$, in (1.6), is then the one given in (1.7). The function $A_s(t)$ depends thus on the relationship between the first-order parameter $\xi$, and the second-order parameter $\rho_0$, just as provided in (6.12). The characterisation of $B_s(t)$ in (6.11) is slightly more complex, and it is presented, jointly with $A_s(t)$, in Table 2.

**Remark 6.3.** *The results presented in* Table 2 *enable us to fully characterise any model in the aforementioned sub-class of* Hall-Welsh*'s class*:

- *For the* Burr$(\xi, \rho)$ *model with d.f. $F(x) = 1 - \left(1 + x^{-\rho/\xi}\right)^{1/\rho}$ $(x > 0,\ \xi > 0,\ \rho \equiv \rho_0 < 0)$ we have $C = 1$, $D_1 = \xi/\rho_0$ and $D_2 = D_1(1 + D_1)/2$.*

- *For the* Fréchet$(\xi)$ *model with d.f. $F(x) = \exp\left(-x^{-1/\xi}\right)$ $(x > 0,\ \xi > 0)$, we have $\rho_0 = -1$, $C = 1$, $D_1 = -\xi/2$ and $D_2 = D_1\left(5/6 + D_1\right)/2$.*

- *For the* generalised Pareto (GP)$(\xi > 0)$ *model with d.f. $F(x) = 1 - (1 + \xi x)^{-1/\xi}$, $(x > 0,\ \xi > 0)$, we have $\rho_0 = -\xi$, $C = 1/\xi$, $D_1 = -1$ and $D_2 = 0$.*

Table 2: Characterisation of second, third-order parameters and functionals $A_s$ and $B_s$ for a model $F$ in the Hall-Welsh sub-class of models, in (6.13), additionally subject to a shift $s \neq 0$.

| | | $A_s(t) := \xi \beta_s t^{\rho_s}$ | $B_s(t) := \beta'_s t^{\rho'_s}$ |
|---|---|---|---|
| $\xi + \rho_0 < 0$ | $\rho_0 < -2\xi$ | $-\frac{\xi s}{C} t^{-\xi}$ | $-\frac{s}{C} t^{-\xi}$ |
| | $\rho_0 = -2\xi$ | | $\left(\frac{2\xi\beta_0 C}{\rho_0 s} - \frac{s}{C}\right) t^{-\xi}$ |
| | $\rho_0 > -2\xi$ | | $-\frac{\beta_0 C}{s}\, t^{\xi+\rho_0}$ |
| $\xi + \rho_0 = 0$ | | $\left(\xi\beta_0 + \frac{\rho_0 s}{C}\right) t^{\rho_0}$ | $\frac{\beta_0 C^2(\beta'_0 - \beta_0) + 2(\beta_0 C - s)^2}{2C(\beta_0 C - s)} t^{\rho_0}$ |
| $\xi + \rho_0 > 0$ | $\rho_0 < -\xi/2$ | $\xi\beta_0 t^{\rho_0} \equiv A_0(t)$ | $-\frac{s}{\beta_0 C} t^{-(\xi+\rho_0)}$ |
| | $\rho_0 = -\xi/2$ | | $\left(\beta'_0 + \frac{2\rho_0 s}{\xi\beta_0 C}\right) t^{\rho_0}$ |
| | $\rho_0 > -\xi/2$ | | $\beta'_0 t^{\rho_0} \equiv B_0(t)$ |

- *The* Student*'s-$t_\nu$ $(\nu > 0)$ distribution is*

$$F(x) = F(x|\nu) = \frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)\sqrt{\pi\nu}} \int_{-\infty}^{x} \left(1 + \frac{z^2}{\nu}\right)^{-(\nu+1)/2} dz, \;\; x \in \mathbb{R}, \;\; \nu > 0,$$

*with $\xi = 1/\nu$ and $\rho_0 = -2/\nu = -2\xi$. In this case we have, $C = \sqrt{\nu}/c_\nu$, where $c_\nu = (\nu \mathcal{B}(\nu/2, 1/2))^{1/\nu}$, with $\mathcal{B}$ the complete Beta function, and $D_1 = -c_\nu^2(\nu+1)/(2(\nu+2))$. When $\nu = 1$ we get the so called* Cauchy *d.f., $F(x) = 1/2 - (\arctan(x))/\pi$, $x \in \mathbb{R}$, with $\xi = 1$ and $\rho_0 = -2$. For the* Cauchy *distribution we have $C = 1/\pi$ and $D_1 = -\pi^2/3$.*

**Remark 6.4.** *Just as mentioned in* Remark 3.1*, note that we can use in* (6.10)*,* (6.11) *and* (6.12) *the subscript $q$ instead of the subscript $s$, whenever we think on such a shift as $s = -\chi_q$.*

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# References

[1] Araújo Santos, P., Fraga Alves, M.I. and Gomes, M.I. (2006). Peaks over random threshold methodology for tail index and quantile estimation. *Revstat* **4**:3, 227–247.

[2] Beirlant, J., Caeiro, F. and Gomes, M.I. (2012). An overview and open research topics in the field of statistics of univariate extremes. *Revstat* **10**:1, 1-31.

[3] Caeiro, F. and Gomes, M.I. (2006). A new class of estimators of a "scale" second order parameter. *Extremes* **9**, 193–211.

[4] Davison, A. (1984). Modeling excesses over high threshold with an application. In J. Tiago de Oliveira ed., *Statistical Extremes and Applications*, D. Reidel, 461–482.

[5] Dekkers, A., Einmahl, J. and de Haan, L. (1989). A moment estimator for the index of an extreme-value distribution. *Annals of Statistics* **17**, 1833–1855.

[6] Drees, H., Ferreira, A. and de Haan, L. (2004). On maximum likelihood estimation of the extreme value index. *Ann. Appl. Probab.* **14**, 1179–1201.

[7] Fraga Alves, M.I., Gomes, M.I. and de Haan, L. (2003). A new class of semi-parametric estimators of the second order parameter. *Portugaliae Mathematica* **60**:2, 194–213.

[8] Fraga Alves, M. I.; Haan, L. de and Lin, T. (2006). Third order extended regular variation, *Publications de l'Institut Mathématique* **80**:94, 109–120.

[9] Geluk, J. and de Haan, L. (1987). *Regular Variation, Extensions and Tauberian Theorems.* CWI Tract 40, Center for Mathematics and Computer Science, Amsterdam, The Netherlands.

[10] Gnedenko, B.V. (1943). Sur la distribution limite du terme maximum d'une série aléatoire. *Annals of Mathematics* **44**, 423–453.

[11] Gomes, M.I. (2003). Stochastic processes in telecommunication traffic. In Fernandes, C. *et al.* (eds.), *Mathematical Techniques and Problems in Telecommunications*, CIM edition, 7–32.

[12] Gomes, M.I. and Guillou, A. (2014). Extreme Value Theory and Statistics of Univariate Extremes: A Review. *International Statistical Review*, accepted.

[13] Gomes, M.I. and Henriques-Rodrigues, L. (2012). Adaptive PORT-MVRB estimation of the extreme value index. In Oliveira, P., Temido, M.G., Henriques, C. and Vichi M. (eds.). *Studies in Theoretical and Applied Statistics: Subseries B: Recent Developments in Modeling and Applications in Statistics.* Springer, 117–125.

[14] Gomes, M.I. and Martins, M.J. (2002). "Asymptotically unbiased" estimators of the tail index based on external estimation of the second order parameter. *J. Statist. Planning and Inference* **93**, 161–180.

[15] Gomes, M.I., de Haan, L. and Peng, L. (2002). Semi-parametric estimation of the second order parameter—asymptotic and finite sample behaviour. *Extremes* **5**:4, 387–414.

[16] Gomes, M.I., Fraga Alves, M.I. and Araújo Santos, P. (2008). PORT Hill and moment estimators for heavy-tailed models. *Commun. in Statist.—Simul. and Comput.* **37**, 1281–1306.

[17] Gomes, M.I., Figueiredo, F. and Neves, M.M. (2012). Adaptive estimation of heavy right tails: resampling-based methods in action. *Extremes* **15**, 463–489.

[18] Gomes, M.I., Henriques-Rodrigues, L., Fraga Alves, M.I. and Manjunath, B.G. (2013). Adaptive PORT-MVRB estimation: an empirical comparison of two heuristic algorithms. *J. Statist. Comput. Simul.* **83**:6, 1129–1144.

[19] Haan, L. de (1984). Slow variation and characterization of domains of attraction. In Tiago de Oliveira, ed., *Statistical Extremes and Applications*, 31-48, D. Reidel, Dordrecht, Holland.

[20] Hall, P. (1982). On estimating the endpoint of a distribution. *Ann. Statist.* **10**, 556–568.

[21] Hall, P. and Welsh, A.W. (1985). Adaptive estimates of parameters of regular variation. *Annals of Statistics* **13**, 331–341.

[22] Henriques-Rodrigues, L. and Gomes, M.I. (2009). High quantile estimation and the PORT methodology. *Revstat* **7**:3, 245–264.

[23] Henriques-Rodrigues, L., Gomes, M.I., Fraga Alves, M.I. and Neves, C. (2014). PORT-estimation of a shape second-order parameter. *Revstat* **12**:3, in press.

[24] Hüsler, J. (2009). Extreme value analysis in biometrics. *Biometrical Journal* **51**:2, 252–272.

[25] Pickands III, J. (1975). Statistical inference using extreme order statistics. *Ann. Statist.*, **3**, 119–131.

[26] Reiss, R.-D. and Thomas, M. (2001; 2007). *Statistical Analysis of Extreme Values, with Application to Insurance, Finance, Hydrology and Other Fields*, 2nd edition, 3rd edition, Birkhäuser Verlag.

[27] Resnick, S.I. (1997). Heavy tail modelling and teletraffic data. *Annals of Statistics* **25**, 1805–1869.

[28] van der Vaart, A.W. (1998). *Asymptotic Statistics.* Cambridge University Press.

[29] Wang, X. and Cheng, S. (2005). General regular variation of n-th order and the 2nd order edgeworth expansion of the extreme value distribution. *Acta Mathematica Sinica* **21**:5, 1121–1130.