Auto-Regressive Extensions of Uniform Randomness Characterization

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Abstract: General results on the algebra of random variables $X_m,\ m \in [-2,2]$, whose distribution is a $1-\frac{|m|}{2}$ mixture of uniform with $X_2 \frown Beta(2,1)$ or $X_{-2} \frown Beta(1,2)$ are used to investigate an autoregressive model that can be used to test uniformity and/or non-correlation, an important issue when combining p-values to asses a composite hypothesis on the validity of the null hypothesis when meta analysing independent tests.

Keywords: Uniformity, mixtures of Uniform and Beta(1,2) or Beta(2,1), auto-regressive sequence, combined p-values, computationally augmented samples.

1 Introduction

Let U and X be two independent random variables, $U \frown Uniform(0,1)$ and X with support $[0,1]$. Denoting by f_X the probability density function (pdf) of X, the random variable

$$V = \min\left(\frac{U}{X}, \frac{1-U}{1-X}\right)$$

has support $(0,1)$, and for $v \in (0,1)$

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$$\mathbb{P}\left[\min\left(\frac{U}{X},\,\frac{1-U}{1-X}\right)\leq v\right]=\int_0^1 vxf_X(x)\,\mathrm{d}x+\int_0^1 v(1-x)f_X(x)\,\mathrm{d}x=v.$$

Moreover, for any $x,v\in(0,1)$,

$$\mathbb{P}\left[\min\left(\frac{U}{X},\,\frac{1-U}{1-X}\right)\leq v|X=x\right]=xv+(1-x)v=\mathbb{P}\left[\min\left(\frac{U}{X},\,\frac{1-U}{1-X}\right)\leq v\right]$$

and therefore $V\frown Uniform(0,1)$, and X and V are independent.

The above result is an appealing form of generating uniform pseudo-random numbers based in previous available random numbers, since surprisingly W and X are independent, and hence if we have a sample of uniform p-values $p=(p_1,...,p_n)$, we can inflate it with new pseudo-p-values $p_{n+k}=\min\left(\frac{u_k}{p_k},\,\frac{1-u_k}{1-p_k}\right)$ using pseudo-random numbers $u_k,\ k=1,...,n$. Thus, in the context of meta analysing results by combining p-values [8, 9], the doubled-size sample $(p_1,...,p_{2n})$, under the validity of H_0, is still the observed value of a uniform random sample.

Artificially augmenting the sample size using the above and connected results cf. [5] consistently produced loss of power of the most usual combined tests, cf. [5, 2, 3].

The issue has been investigated in depth by Brilhante *et al*. [1, 3] (cf. also Hartung *et al*. [6] and Kulinskaya *et al*. [7] on the concepts of generalised p-values and of random p-values) since when meta analysing results from independent tests via combing p-values, assuming that the null hypothesis is true in all cases is farfetched, and whenever the alternative hypothesis is true the observed p-value is no longer the observation of an uniform random variable.

Brilhante *et al*. [3] investigated in detail the family of "tilted" uniform densities with common point $(0.5,1)$. For $m\in[-2,2]$, the probability density functions

$$f_m(x)=[1+m\,(x-0.5)]\,\mathbb{I}_{(0,1)}(x) \tag{1}$$

is the convex combination of $f_0(x)=\mathbb{I}_{(0,1)}(x)$ — the standard uniform density — and of $f_{-2}(x)=2(1-x)\,\mathbb{I}_{(0,1)}(x)$ if $m\in[-2,0)$, or of $f_2(x)=2x\,\mathbb{I}_{(0,1)}(x)$ if $m\in(0,2]$. In either case, the mixing coefficient of the uniform is $1-\frac{|m|}{2}$. In other words, we are considering random variables $X_m,\ m\in[-2,2]$,

$$X_m=\left\{\begin{array}{cc} U & X_{\frac{2|m|}{m}} \\ & \\ 1-\frac{|m|}{2} & \frac{|m|}{2}\end{array}\right. \tag{2}$$

that can represent more proneness to obtain small p-values near 0 if $m\in[-2,0)$, or more proneness to obtain large p-values near 1 if $m\in(0,2]$.

The interesting point is that if X_{m_1},X_{m_2} are independent random variables with pdf $f_{m_i}(x)=\left[1+m_i\left(x-\frac{1}{2}\right)\right]\,\mathbb{I}_{(0,1)}(x)$, with $m_i\in[-2,2],\ i=1,2$, then

$$V_{m_1,m_2}=\min\left(\frac{X_{m_1}}{X_{m_2}},\frac{1-X_{m_1}}{1-X_{m_2}},\right)\ =\ X_{\frac{m_1\,m_2}{6}}=\left\{\begin{array}{cc} U & X_{\frac{2|m_1\,m_2|}{m_1\,m_2}} \\ & \\ 1-\frac{|m_1\,m_2|}{12} & \frac{|m_1\,m_2|}{12}\end{array}\right..$$

We shall use this result to discuss testing uniformity of auto-regressive sequences generated by independent uniform sequences and, as a side result, independence.

2 Auto-Regressive Model Generated by Sequences of Independent Uniform Random Variables

Let $U_0, U_1, \dots$ be independent and identicaly distributeted (iid) standard uniform random variables (rv's) and consider the autoregressive model

$$X_k = \rho\, X_{k-1} + (1-\rho)\, U_k, \quad \rho \in (0,1),\; k = 1,2,\dots$$

It is immediate that

$$\min\left\{\frac{X_k}{X_{k-1}}, \frac{1-X_k}{1-X_{k-1}}\right\} = \min\left\{\frac{\rho\, X_{k-1} + (1-\rho)\, U_k}{X_{k-1}}, \frac{\rho + 1 - \rho - \rho\, X_{k-1} - (1-\rho)\, U_k}{1-X_{k-1}}\right\} =$$

$$= \rho + (1-\rho)\, \min\left\{\frac{U_k}{X_{k-1}}, \frac{1-U_k}{1-X_{k-1}}\right\} = \rho + (1-\rho)\, U.$$

Therefore, from the results in Section 1,

$$\min\left\{\frac{X_k}{X_{k-1}}, \frac{1-X_k}{1-X_{k-1}}\right\} \frown Uniform(\rho, 1).$$

3 Auto-Regressive Model Generated by Sequences of Independent X_m random variables

We now investigate the more general setting

$$\min\left\{\frac{Y_k}{Y_{k-1}}, \frac{1-Y_k}{1-Y_{k-1}}\right\}$$

where

$$Y_k = \rho\, Y_{k-1} + (1-\rho)\, X_{m,k},\; \rho \in (0,1),\; k = 1,2,\dots$$

with $X_{m,k},\; k = 0,1,\dots,$ independent replicas of X_m as defined in (2).

Let $\{X_{m,i}\},\; i \geq 0$, be a sequence of independent replicas of $X_m,\; m \in [-2,2]$, i.e., independent random variables with pdf (1).

Define

$$Y_{m,i} = \rho\, Y_{m,i-1} + (1-\rho)\, X_{m,i}, \;\; Y_{m,0} = X_{m,0},$$

$1 \leq i \leq n,\; \rho \in [0,1)$.

If $\rho = 0$, the $\{Y_{m,i}\},\; i \geq 0$, sequence is the original iid sequence. On the other hand, if $\rho > 0$ serial correlation does exist.

Observe, once again, that for $m = 0$ we have the situation of uniformity, and therefore the above setting is appropriate to test independence of the P_k's, versus serial correlation, under the null hypothesis.

As the inverse transformation is

$$X_{m,i} = \frac{Y_{m,i} - \rho\, Y_{m,i-1}}{1-\rho}, 1 \leq i \leq n,$$

the Jacobian of the inverse transformation is $J=\left(\frac{1}{1-\rho}\right)^n$.

Therefore

$$f_{Y_1,\dots,Y_n}(y_1,\dots,y_n)=\prod_{i=1}^{n}\left(m\,\frac{y_{m,i}-\rho\,y_{m,i-1}}{1-\rho}+\frac{2-m}{2}\right)\frac{1}{1-\rho}\,\mathbb{I}_S(y_1,...,y_n),$$

where $S=\bigcap_{i=1}^{n}\left\{0<\dfrac{y_{m,i}-\rho\,y_{m,i-1}}{1-\rho}<1\right\}$.

Observing that

$$\bigcap_{i=1}^{n}\left\{0<\frac{y_{m,i}-\rho\,y_{m,i-1}}{1-\rho}<1\right\}\Longleftrightarrow\rho\leq\min_{1\leq i\leq n}\left\{\frac{y_{m,i}}{y_{m,i-1}},\frac{1-y_{m,i}}{1-y_{m,i-1}}\right\},$$

for a standard uniform generating sequence ($m=0$), the joint pdf of $Y_1,\dots,Y_n$ is

$$f_{Y_1,\dots,Y_n}(y_1,\dots,y_n)=\prod_{i=1}^{n}\frac{1}{1-\rho}\,\mathbb{I}_{\rho\leq\min_{1\leq i\leq n}\left\{\frac{y_{0,i}}{y_{0,i-1}},\frac{1-y_{0,i}}{1-y_{0,i-1}}\right\}}.$$

Hence, in the special case $m=0$ (uniformity of the generating sequence), and under the null hypothesis $H_0:\ \rho=0$ of independence we shall have to deal with

$$\min_{1\leq i\leq n}\min\left\{\frac{X_{0,i}}{X_{0,i-1}},\frac{1-X_{0,i}}{1-X_{0,i-1}}\right\}=\min_{1\leq i\leq n}\{U_1,\dots,U_n\},$$

where $\{U_1,\dots,U_n\}$ is a sequence of iid standard uniform random variables, and hence

$$\min_{1\leq i\leq n}\{U_1,\dots,U_n\}\frown Beta(1,n).$$

More generally, without assuming $\rho=0$, but still with $m=0$,

$$\min\left\{\frac{Y_{0,i}}{Y_{0,i-1}},\frac{1-Y_{0,i}}{1-Y_{0,i-1}}\right\}=\min\left\{\rho+(1-\rho)\frac{X_{0,i}}{X_{0,i-1}},\rho+(1-\rho)\frac{1-X_{0,i}}{1-X_{0,i-1}}\right\}$$

is uniform in $(\rho,1]$.

Denote by $\min\left\{\rho+(1-\rho)\frac{X_{0,i}}{X_{0,i-1}},\rho+(1-\rho)\frac{1-X_{0,i}}{1-X_{0,i-1}}\right\}=V_{i,\rho}$.

$V=\min_{1\leq i\leq n}V_{i,\rho}$ is the MLE of ρ, and it is sufficient for ρ. The likelihood function — that is also the likelihood ratio for testing $H_0:\rho=0$ *vs.* $H_A:\rho>0$ — is

$$L(\rho)=\left(\frac{1}{1-\rho}\right)^n\mathbb{I}_{\rho\leq V}.$$

A most powerful α-test rejects the null hypothesis when $V>1-\alpha^{1/n}$, power being given by

$$\begin{cases}\dfrac{\alpha}{(1-\rho)^n} & \text{if}\rho\leq1-\alpha^{1/n}\\ 1 & \text{otherwise}\end{cases}.$$

Next, observe that for general $m \in [-2,2]$,

$$\frac{Y_{m,i}}{Y_{m,i-1}} = \rho + (1-\rho)\,\frac{X_{m,i}}{X_{m,i-1}}$$

and

$$\frac{1-Y_{m,i}}{1-Y_{m,i-1}} = \rho + (1-\rho)\,\frac{1-X_{m,i}}{1-X_{m,i-1}}$$

so that

$$\min\left(\frac{Y_{m,i}}{Y_{m,i-1}},\frac{1-Y_{m,i}}{1-Y_{m,i-1}}\right) = \rho + (1-\rho)\,\min\left(\frac{X_{m,i}}{X_{m,i-1}},\frac{1-X_{m,i}}{1-X_{m,i-1}}\right),$$

and from the independence of the $X_{m,i}$'s it follows that

$$\min\left(\frac{Y_{m,i}}{Y_{m,i-1}},\frac{1-Y_{m,i}}{1-Y_{m,i-1}}\right) \stackrel{\mathrm{d}}{=} \rho + (1-\rho)\,X_{\frac{m^2}{6}}\,.$$

Hence our interest now lies on the minimum of ρ-shifted and $(1-\rho)$-rescaled $X_{\frac{m^2}{6}}$ random variables. Once again this minimum is the MLE, and sufficient, for ρ.

Small sample exact results are cumbersome, but simulation results can be dealt with easily, and approximations using an asymptotic rationale quite satisfactory, since the convergence rate is high.

Appendix: Autoregressive Models generated by uniforms

Consider, to start with, the order 1 auto-regressive model

$$Y_n = \rho Y_{n-1} + (1-\rho)U_n\,, \quad n = 1,2,\ldots$$

where the U_n, $n = 0,1,2,\ldots$ are independent replicas of $U \frown Uniform(0,1)$ and $Y_0 = U_0$. In Figure 1 we display a possible realization of this model for $\rho = 0.4$ and $n = 500$.

The above model can be re-expressed as

$$Y_n = \rho^n U_0 + (1-\rho)\sum_{k=1}^{n}\rho^{n-k}U_k\,, \quad n = 1,2,\ldots$$

and hence

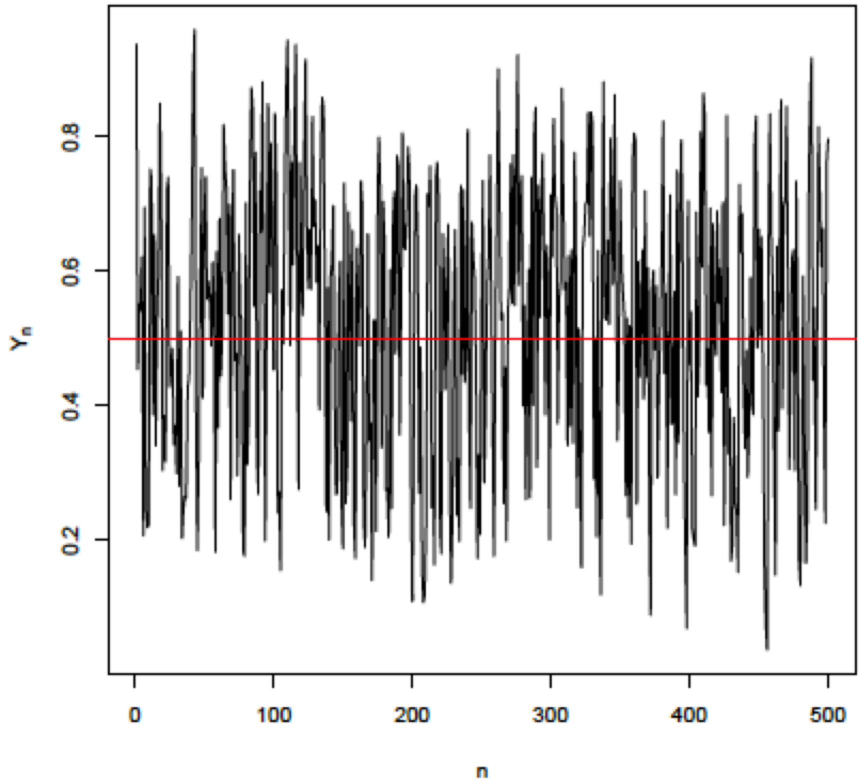
$$\mathbb{E}(Y_n) = \frac{\rho^n}{2} + \frac{1-\rho}{2}\sum_{k=1}^{n}\rho^{n-k} = \frac{1}{2}.$$

Since the expectation of Y_n does not depend on n, this is a stationary series.

On the other hand, the joint pdf of (Y_{n-1},U) is

$$f_{(Y_{n-1},U)}(y_{n-1},u) = f_{Y_{n-1}}(y_{n-1})\mathrm{I}_A(y_{n-1},u),$$

Fig. 1 $Y_n = 0.4\,Y_{n-1} + 0.6\,U_n$, generated for $n = 500$



where $A = \{(y_{n-1}, u) : 0 < y_{n-1} < 1, 0 < u < 1\}$. Thus, for the transformation $Y_{n-1} = Y_{n-1}$ and $Y_n = \rho Y_{n-1} + (1-\rho)U$, with jacabioan $J = \frac{1}{1-\rho} > 0$, we obtain for the joint pdf of (Y_{n-1}, Y_n)

$$f_{(Y_{n-1},Y_n)}(y_{n-1}, y_n) = \frac{f_{Y_{n-1}}(y_{n-1})}{1-\rho} \mathrm{I}_B(y_{n-1}, y_n)$$

where $B = \{(y_{n-1}, y_n) : 0 < y_{n-1} < 1, \rho y_{n-1} < y_n < 1 - \rho + \rho y_{n-1}\}$.

Therefore,

$$f_{Y_n}(y_n) = \int_{\mathbb{R}} f_{(Y_{n-1},Y_n)}(y_{n-1}, y_n) dy_{n-1}, \quad n = 1, 2, \ldots$$

An explicit evaluation for $n = 1, 2, 3$ is feasible:

- $n = 1$

$$f_{Y_1}(y) = \begin{cases} \frac{y}{\rho(1-\rho)} & , y \in]0, \rho] \\ \frac{1}{1-\rho} & , y \in]\rho, 1-\rho] \\ \frac{1-y}{\rho(1-\rho)} & , y \in]1-\rho, 1[\\ 0 & , y \notin]0, 1[\end{cases} .$$

- $n = 2$

$$f_{Y_2}(y) = \begin{cases} \frac{y^2}{2\rho^3(1-\rho)^2} & , y \in]0, \rho^2] \\ \frac{2y-\rho^2}{2\rho(1-\rho)^2} & , y \in]\rho^2, \rho(1-\rho)] \\ \frac{-y^2+2\rho y-2\rho^4+2\rho^3-\rho^2}{2\rho^3(1-\rho)^2} & , y \in]\rho(1-\rho), \rho] \\ \frac{1}{1-\rho} & , y \in]\rho, 1-\rho] \\ \frac{-(1-y)^2+2\rho(1-y)-2\rho^4+2\rho^3-\rho^2}{2\rho^3(1-\rho)^2} & , y \in]1-\rho, 1-\rho(1-\rho)] \\ \frac{2(1-y)-\rho^2}{2\rho(1-\rho)^2} & , y \in]1-\rho(1-\rho), 1-\rho^2] \\ \frac{(1-y)^2}{2\rho^3(1-\rho)^2} & , y \in]1-\rho^2, 1[\\ 0 & , y \notin]0, 1[\end{cases}$$

- $n = 3$

$$f_{Y_3}(y) = \begin{cases} \frac{y^3}{6\rho^6(1-\rho)^3} & , y \in]0, \rho^3] \\ \frac{3y^2-3\rho^3 y+\rho^6}{6\rho^3(1-\rho)^3} & , y \in]\rho^3, \rho^2(1-\rho)] \\ \frac{-y^3+3\rho^2y^2+(-6\rho^6+6\rho^5-3\rho^4)y+3\rho^8-3\rho^7+\rho^6}{6\rho^6(1-\rho)^3} & , y \in]\rho^2(1-\rho), \rho^2] \\ \frac{2y-\rho^2}{2\rho(1-\rho)^2} & , y \in]\rho^2, \rho(1-\rho)] \\ \frac{-y^3-3(\rho^2-\rho)y^2-(6\rho^6-6\rho^5+3\rho^4-6\rho^3+3\rho^2)y+3\rho^8-3\rho^7-\rho^6+3\rho^5-3\rho^4+\rho^3}{6\rho^6(1-\rho)^3} & , y \in]\rho(1-\rho), \rho(1-\rho(1-\rho))] \\ \frac{-3y^2-(3\rho^3-6\rho)y-\rho^6+6\rho^5-9\rho^4+6\rho^3-3\rho^2}{6\rho^3(1-\rho)^3} & , y \in]\rho(1-\rho(1-\rho)), \rho(1-\rho^2)] \\ \frac{y^3-3\rho y^2+3\rho^2 y+6\rho^8-12\rho^7+6\rho^6-\rho^3}{6\rho^6(1-\rho)^3} & , y \in]\rho(1-\rho^2), \rho] \\ \frac{1}{1-\rho} & , y \in]\rho, 1-\rho] \\ \frac{(1-y)^3-3\rho(1-y)^2+3\rho^2(1-y)+6\rho^8-12\rho^7+6\rho^6-\rho^3}{6\rho^6(1-\rho)^3} & , y \in]1-p, 1-\rho(1-\rho^2)] \\ \frac{-3(1-y)^2-(3\rho^3-6\rho)(1-y)-\rho^6+6\rho^5-9\rho^4+6\rho^3-3\rho^2}{6\rho^3(1-\rho)^3} & , y \in]1-\rho(1-\rho^2), 1-\rho(1-\rho(1-\rho))] \\ \frac{-(1-y)^3-3(\rho^2-\rho)(1-y)^2-(6\rho^6-6\rho^5+3\rho^4-6\rho^3+3\rho^2)(1-y)+3\rho^8-3\rho^7-\rho^6+3\rho^5-3\rho^4+\rho^3}{6\rho^6(1-\rho)^3} & , y \in]1-\rho(1-\rho(1-\rho)), 1-\rho(1-\rho)] \\ \frac{2(1-y)-\rho^2}{2\rho(1-\rho)^2} & , y \in]1-\rho(1-\rho), 1-\rho^2] \\ \frac{-(1-y)^3+3\rho^2(1-y)^2+(-6\rho^6+6\rho^5-3\rho^4)(1-y)+3\rho^8-3\rho^7+\rho^6}{6\rho^6(1-\rho)^3} & , y \in]1-\rho^2, 1-\rho^2(1-\rho)] \\ \frac{3(1-y)^2-3\rho^3(1-y)+\rho^6}{6\rho^3(1-\rho)^3} & , y \in]1-\rho^2(1-\rho), 1-\rho^3] \\ \frac{(1-y)^3}{6\rho^6(1-\rho)^3} & , y \in]1-\rho^3, 1[\\ 0 & , y \notin]0, 1[\end{cases}$$

The general order n auto-regressive model induced by an iid random sequence is

$$Y_n = \rho_1 Y_{n-1} + \rho_2 Y_{n-2} + \ldots + \rho_k Y_{n-k} + \phi U_n$$

where $\phi = 1 - \sum_{i=1}^{k} \rho_i$. In Section 3 we have established that $\min\left\{\frac{Y_k}{Y_{k-1}}, \frac{1-Y_k}{1-Y_{k-1}}\right\} \frown Uniform(\rho, 1)$. Now define the random variables

$$T_{n,j} = \rho_j + \min\left(\frac{\sum_{i=1,i\neq j}^{k} \rho_i Y_{n-i} + \phi U_n}{Y_{n-j}}, \frac{\sum_{i=1,i\neq j}^{k} \rho_i (1 - Y_{n-i}) + \phi(1 - U_n)}{1 - Y_{n-j}}\right), \quad j = 1, ..., n-1.$$

For the order 2 auto-regressive model some (messy) algebra may be executed:

$$Y_n = \rho_1 Y_{n-1} + \rho_2 Y_{n-2} + \phi U_n\,, \quad n \geq 2$$

in which Y_0, Y_1, U_2 are iid random variables (U_n independent from Y_{n-1} e Y_{n-2}), and

$$\begin{aligned} T_{2,1} &= \min\left(\frac{Y_2}{Y_1}, \frac{1-Y_2}{1-Y_1}\right) = \rho_1 + \min\left(\frac{\rho_2 Y_0 + \phi U_2}{Y_1}, \frac{\rho_2(1-Y_0)+\phi(1-U_2)}{1-Y_1}\right) \\ &= \rho_1 + \min\left(\frac{\rho_2 Y_0 + \phi U}{Y_1}, \frac{\rho_2(1-Y_0)+\phi(1-U)}{1-Y_1}\right) \end{aligned}$$

Denoting by $W = \min\left(\frac{\rho_2 Y_0+\phi U}{Y_1}, \frac{\rho_2(1-Y_0)+\phi(1-U)}{1-Y_1}\right)$,

$$\begin{aligned} \mathbb{P}(W \le w) &= 1 - \mathbb{P}\left(\frac{\rho_2 Y_0 + \phi U}{Y_1} > w, \frac{\rho_2(1-Y_0)+\phi(1-U)}{1-Y_1} > w\right) \\ &= 1 - \mathbb{P}(wY_1 < \rho_2 Y_0 + \phi U < \rho_2 + \phi - w + wY_1). \end{aligned} \tag{3}$$

The random variables $Z = \rho_2 Y_0 + \phi U$ and Y_1 are independent, and in case Y_0 e U are members from the family of random variables with pdf (1), we obtain

$$f_Z(z) = \begin{cases} \frac{3(m-2)^2\rho_2\phi z - 3(m-2)m(\rho_2+\phi)z^2 + 2m^2 z^3}{12\rho_2^2\phi^2} & , z \in (0,a) \\ \frac{12mz - 6m(\rho_2+\phi) - m^2 a + 12b}{12b^2} & , z \in [a,b) \\ \frac{(\rho_2+\phi-z)\left[12\rho_2\phi - 6m(\rho_2^2+\phi^2-(\rho_2+\phi)z)) - m^2(\rho_2^2-\rho_2\phi+\phi^2+(\rho_2+\phi)z-2z^2)\right]}{12\rho_2^2\phi^2} & , z \in [b, \rho_2+\phi) \\ 0 & , z \notin (0, \rho_2+\phi) \end{cases}$$

where $a = \min(\rho_2, \phi)$ and $b = \max(\rho_2, \phi)$.

If $m = 0$,

$$f_Z(z) = \begin{cases} \frac{z}{\phi\rho_2} & , z \in (0,a) \\ \frac{1}{b} & , z \in [a,b) \\ \frac{\rho_2+\phi-z}{\phi\rho_2} & , z \in [b,\rho_2+\phi) \\ 0 & , z \notin (0,\rho_2+\phi) \end{cases}.$$

The computation of (3) is performed considering that $m = 0$ and $f_1(z) = \frac{z}{\phi\rho_2}$, $f_2(z) = \frac{1}{\phi}$ e $f_3(z) = \frac{\rho_2+\phi-z}{\phi\rho_2}$ (with no loss of generality, we assume $\max(\rho_2,\phi) = \phi$).

Under the above provisos,

$$f_{(Y_1,Z)}(y_1,z) = f_{Y_1}(y_1) f_Z(z) = f_Z(z), \quad 0 < y_1 < 1, 0 < z < \rho_2 + \phi,$$

and therefore the distribution function of W is

$$\mathbb{P}(W \le w) = \begin{cases} 0 & , w \le 0 \\ \frac{w^2}{3\phi\rho_2} & , w \in (0,\rho_2) \\ 1 + \frac{\rho_2^2 - 3\rho_2 w + 3w(w-\phi)}{3\phi w} & , w \in [\rho_2, \phi) \\ 1 + \frac{\rho_2^3 + (\phi-w)^3 - 3\rho_2^2 w + 3\rho_2 w(w-\phi)}{3\phi\rho_2 w} & , w \in [\phi, \rho_2+\phi) \\ 1 & , w \ge \rho_2+\phi \end{cases},$$

from which follows the pdf

$$f_W(w) = \begin{cases} \frac{2w}{3\phi\rho_2} & , w \in (0,\rho_2) \\ \frac{3w^2-\rho_2^2}{3\phi w^2} & , w \in [\rho_2,\phi) \\ \frac{3\rho_2 w^2-\rho_2^3-(\phi-w)^2(\phi+2w)}{3\phi\rho_2 w^2} & , w \in [\phi,\rho_2+\phi) \\ 0 & , w \notin (0,\rho_2+\phi) \end{cases}.$$

As $T_{2,1} = \rho_1 + W$, the support of $T_{2,1}$ lies in the interval $(\rho_1, 1)$ and thus

$$f_{T_{2,1}}(t) = \begin{cases} \frac{2(t-\rho_1)}{3\phi\rho_2} & , t \in (\rho_1,\rho_1+\rho_2) \\ \frac{3(t-\rho_1)^2-\rho_2^2}{3\phi(t-\rho_1)^2} & , t \in [\rho_1+\rho_2,\rho_1+\phi) \\ \frac{3\rho_2(t-\rho_1)^2-\rho_2^3-(\phi+\rho_1-t)^2(\phi-2\rho_1+2t)}{3\phi\rho_2(t-\rho_1)^2} & , t \in [\rho_1+\phi,1) \\ 0 & , t \notin (\rho_1,1) \end{cases}.$$

In what concerns the general case $m \in [-2,2]$,

- If $w \in (0,\rho_1)$,

$$\mathbb{P}(W > w) = \frac{60\rho_2^2\phi^2 + m^2(\phi - w)w^3 + \rho_2 w^2(-20\phi + m^2 w)}{60\rho_2^2\phi^2}$$

- If $w \in [\rho_2,\phi)$,

$$\mathbb{P}(W > w) = \frac{m^2(\rho_2 + \phi - w)(\rho_2^3 - 5\rho_2 w^2 + 5w^3) - 20\phi w[\rho_2^2 - 3\rho_2 w + 3w(w-\phi)]}{60\phi^2 w}$$

- If $w \in [\phi,\rho_2+\phi)$,

$$\mathbb{P}[W > w] = \frac{(\rho_2 + \phi - w)\left\{-20\rho_2\phi w[\rho_2^2 + (\phi - w)^2 - \rho_2(\phi + 2w)]\right\}}{60\rho_2^2\phi^2 w^2}.$$

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