Extensions of the Verhulst Model, Order Statistics and Products of Independent Uniform Random Variables

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Abstract. Several extensions of the Verhulst sustainable population growth model exhibit different interesting characteristics more appropriate to deal with less controlled population dynamics. As the logistic parabola $x(1-x)$ arising in the Verhulst differential equation is closely related to the Beta(2,2) probability density, and the retroaction factor $1-x$ is the linear truncation of MacLaurin series of $-\ln x$ (the growth factor x is the linear truncation of $-\ln(1-x)$), in previous papers the authors introduced a more general four parameter family of probability density functions, of which the classical Beta densities are special cases. Using differential equations extending the original Verhulst, they have been able to identify combinations of parameters that lead to extreme value models, either for maxima or for minima, and also remarked that the traditional logistic model is a (geometric) extreme value model arising from geometric thinning of the original sequence. The observation that in the support $(0,1)$ the logistic parabola $x(1-x)$ is, up to a multiplicative factor, the product of the densities of minimum and maximum of two standard independent uniform random variables (and also the median of three independent standard uniforms), and that on the other hand $(-\ln x)^{n-1}$ is, up to the multiplicative factor $1/\Gamma(n)$, the density of the product of n independent uniforms, we reexamine the ties of products and of order statistics of independent uniforms to dynamical properties of populations arising in these extensions of the Verhulst model.
Keywords: Extended Verhulst models, instabilities in population dynamics, products and order statistics of uniform random variables.

1 Extensions of the Verhulst Model

Extensions of the classical Verhulst differential equation for modeling population dynamics

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = rN(t)(1-N(t)), \tag{1}$$

where $N(t)$ denotes the size of the population at time t and $r > 0$ is the malthusian reproduction rate, have recently been considered.

From the fact that the logistic parabola $x(1-x)$ arising from equation (1) is, in the support $(0,1)$, closely tied to the Beta(2,2) probability density function (pdf)[1], natural extensions of equation (1) using more general beta densities have been investigated by Aleixo *et al.* [1] and Pestana *et al.* [5], namely by considering the differential equation

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = r(N(t))^{p-1}(1-N(t))^{q-1}\,. \tag{2}$$

The normalized solution of equation (1) belongs to the family of logistic functions, which is connected to extreme value models, more precisely to max-geo-stable laws, and occurring in randomly stopped extremes schemes with geometric subordinator. On the other hand, Aleixo *et al.* [1] showed that the normalized solution of equation (2) also belongs to the class of max-geo-stable laws if $p = 2-\alpha$ and $q = 2+\alpha$ (the classical Verhulst model being the special case $\alpha = 0$).

By noticing that the retroaction factor $1-x$ in the logistic parabola is the linear truncation of MacLaurin series of $-\ln x$, and that the growth factor x is the linear truncation of MacLaurin series of $-\ln(1-x)$, Brilhante *et al.* [2] introduced a general four parameter family of densities, named the BeTaBoOp family, which was used to further extend equation (2) in Brilhante *et al.* [2] and [4].

Definition. A random variable X is said to have a BeTaBoOp(p,q,P,Q) distribution, $p,q,P,Q > 0$, if its pdf is

$$f(x) = kx^{p-1}(1-x)^{q-1}(-\ln(1-x))^{P-1}(-\ln x)^{Q-1}\mathrm{I}_{(0,1)}(x)\,, \tag{3}$$

where $k^{-1} = \int_0^1 t^{p-1}(1-t)^{q-1}(-\ln(1-t))^{P-1}(-\ln t)^{Q-1}\mathrm{d}t$ (Hölder's inequality guarantees that $k^{-1} < \infty$).

Note that the Beta(p,q) density is the BeTaBoOp$(p,q,1,1)$ density and if $q = P = 1$, the Betinha(p,Q) density introduced by Brilhante *et al.* [3] is obtained, where $k = \frac{p^Q}{\Gamma(Q)}$ and $\Gamma(\alpha) = \int_0^1 t^{\alpha-1}\mathrm{e}^{-t}\mathrm{d}t$ is the gamma function.

Hence, for a general discussion of growth models, it seems interesting to investigate the general differential equation

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = r(N(t))^{p-1}(1-N(t))^{q-1}[-\ln(1-N(t))]^{P-1}(-\ln N(t))^{Q-1}\,, \tag{4}$$

specially for the case when some of the parameters take the value 1.

Exact solutions for equation (4) exist for some special combinations of the parameters, and when solving the corresponding difference equation

$$x_{t+1} = c\,(x_t)^{p-1}(1-x_t)^{q-1}(-\ln(1-x_t))^{P-1}(-\ln x_t)^{Q-1}$$

[1] A random variable X is said to have a Beta(p,q) distribution, $p,q > 0$, if its pdf is $f(x) = \frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}\mathrm{I}_{(0,1)}(x)$, where $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\mathrm{d}t$ is the Beta function.

by the fixed point method, bifurcation and chaos are observed (see Brilhante *et al.* [2] and [4]).

2 Understanding population dynamics through order statistics and products of powers of uniform random variables

In section 1 we saw that the Verhulst differential equation and extensions are linked to BeTaBoOp densities. Using the fact that these densities can be expressed as functions of densities of order statistics and/or products of independent standard uniform random variables, we reexamine in this section the dynamical properties of populations described by the Verhulst model and extensions.

Let $U_1, \ldots, U_n$ be independent and identically distributed (iid) standard uniform random variables, and let $U_n^{(*)}$ denote their product, whose pdf is

$$f_{U_n^{(*)}}(u) = \frac{(-\ln u)^{n-1}}{\Gamma(n)} \mathrm{I}_{(0,1)}(u)\,. \tag{5}$$

The pdf (5) is easily derived by simply noting that $U_n^{(*)} = \exp(-V)$, with $V = -\sum_{i=1}^n \ln U_i = -\ln\left(\prod_{i=1}^n U_i\right) \frown \mathrm{Gamma}(n,1)$. More generally, $U_n^{\delta^{(*)}} = \left(\prod_{i=1}^n U_i\right)^\delta = \prod_{i=1}^n U_i^\delta$, $\delta > 0$, has pdf

$$f_{U_n^{\delta^{(*)}}}(u) = \frac{u^{1/\delta - 1}(-\ln u)^{n-1}}{\delta^n \Gamma(n)} \mathrm{I}_{(0,1)}(u)$$

and distribution function

$$F_{U_n^{\delta^{(*)}}}(u) = \frac{\Gamma(n, -\ln u/\delta)}{\Gamma(n)} = u^{1/\delta} \sum_{k=0}^{n-1} \frac{(-\ln u)^k}{\delta^k k!}\,,\ u \in (0,1).$$

On the other hand, let $U_{k:n}$ denote the k-th ascending order statistic, $k = 1, \ldots, n$, whose pdf is

$$f_{U_{k:n}}(u) = \frac{u^{k-1}(1-u)^{n-k}}{B(k, n+1-k)} \mathrm{I}_{(0,1)}(u)\,,$$

i.e. $U_{k:n} \frown \mathrm{Beta}(k, n+1-k)$. In particular, the minimum $U_{1:n}$ has pdf $f_{U_{1:n}}(u) = n(1-u)^{n-1}\,\mathrm{I}_{(0,1)}(u)$, and the maximum $U_{n:n}$ pdf $f_{U_{n:n}}(u) = nu^{n-1}\,\mathrm{I}_{(0,1)}(u)$.

For the special case $n = 2$, it is obvious that $U_1 U_2 = U_{1:2} U_{2:2} \preceq U_{1:2} \preceq U_{2:2}$, and a similar result holds true for all $n \in \mathbb{N}$, $n > 2$.

Thus, when $p, q, P, Q \in \mathbb{N}$, the pdf of the BeTaBoOp(p, q, P, Q) random variable is, up to a multiplicative factor, the product of the densities of the maximum $U_{p:p}$ of p independent standard uniforms, of the minimum $U_{1:q}$ of q independent standard uniform random variables, of the product $U_P^{(*)}$ of P

independent standard uniform random variables, and of $1-U_Q^{(*)}$. Observe also that in the long-standing established jargon of population dynamics, the x^{p-1} and $(-\ln(1-x))^{P-1}$ are growing factors, and $(1-x)^{q-1}$ and $(-\ln x)^{Q-1}$ are retroaction factors, curbing down population growth. In view of the above remarks on the connection to ascending order statistics and products of independent standard uniform random variables, we shall say that $(-\ln x)^{\nu-1}$ is a lighter retroaction factor than $(1-x)^{\nu-1}$, and that $(-\ln(1-x))^{\mu-1}$ is a heavier growth factor than $x^{\mu-1}$.

In view of the facts above, it is expectable that the normalized solution of the differential equation linked to the Betinha(2,2) $\equiv$ BeTaBoOp(2,1,1,2) density, which can be obtained by replacing in (1) the retroaction factor $1-N(t)$ by the lighter one $-\ln N(t)$, will correspond to less sustainable growth.

In fact, the solution of that differential equation is the Gompertz function, that up to a multiplicative factor is the extreme value Gumbel distribution. Note that while the logistic distribution is a stable limit law for suitably linearly modified maxima of geometrically thinned sequences of iid random variables in its domain of attraction is known to be appropriate to model sustainable growth, the Gumbel distribution arises as stable limit law of suitably normalized maxima of all the random variables in its domain of attraction[2], and therefore stochastically dominates the logistic solution, and is a suitable model for uncontrolled growth, such as the one observed for cells of cancer tumors.

More generally, Brilhante *et al.* [2] have shown that the normalized solution of the differential equation tied to the more general BeTaBoOp$(2,1,1,2+\alpha)$ density, i.e.

$$\frac{\mathrm{d}N(t)}{\mathrm{dt}} = rN(t)(-\ln N(t))^{1+\alpha}\,, \tag{6}$$

belongs to the class of extreme value laws for maxima, more precisely Gumbel if $\alpha=0$, Fréchet if $\alpha>0$ and Weibull for maxima if $\alpha<0$. Therefore, equation (6) reveals to be more appropriate then (1) to deal with less controlled population dynamics.

On the other hand, if the growth factor $N(t)$ in (1) is replaced by $[-\ln(1-N(t))]^{1+\alpha}$, we get a differential equation linked to the BeTaBoOp$(1,2,2+\alpha,1)$ density, whose normalized solution now belongs to the class of extreme value laws for minima. Using the fact that if $X \frown$ BeTaBoOp(p,q,P,Q), $1-X \frown$ BeTaBoOp(q,p,Q,P), simplifies the investigations concerning the structural properties of the BeTaBoOp family, namely those related to products of uniform random variables.

Therefore, equations (1), (2) and (6) can be viewed as special cases of the more general differential equation (4) for modeling population dynamics, which embodies simultaneously two different growth patterns depicted in the growing terms $(N(t))^{p-1}$ and $[-\ln(1-N(t))]^{P-1}$, and two different environmental resources control of the growth behavior, depicted in the retroaction terms $(1-N(t))^{q-1}$ and $(-\ln N(t))^{Q-1}$.

[2] Note that Rachev and Resnick [6] established a connection between extreme stable laws and geometrically thinned extreme value laws, which implies, in particular, that when they have the same index — 0 in case of the Gumbel and of the logistic stable limits — they share the same domain of attraction.

We obtained explicit solutions for (4), using Mathematica, for a few special combinations of parameters, but so far only the ones connected with some form of stability and of extreme value models — either in the iid setting or in the geometrically thinned setting — seem to be suitable to characterize growth. In the sequel we shall comment on growth characteristics, in general, in terms of the order relation among parameters, and specially when all the parameters are integers.

3 Further comments for the special case of integer parameters

The Verhulst model is usually associated with the idea of sustainable growth. This is the case since the retroaction term $1 - N(t)$ slows down the growth impetus $rN(t)$, an equilibrium often interpreted as sustainability. Another way of looking at this is to notice that the logistic parabola $x(1-x)$ tied to the Verhulst model is, up to a multiplicative factor, the product of the densities of the order statistics $U_{2:2}$ and $U_{1:2}$ — respectively, maximum and minimum of two independent standard uniform random variables. Therefore, the growth term ruled by $U_{2:2}$ has an "equal" opposite effect exerted by the retroaction term ruled by $U_{1:2}$, which is curbing down the population growth to sustainable levels. On the other hand, we also have that the logistic parabola is proportional to the density of $U_{2:3}$, i.e. the median of three independent standard uniform random variables, thus reinforcing the idea of equilibrium.

We now amplify the above remarks to other interesting cases of the generalized Verhulst growth theory:

1. The logistic parabola generalization $x^{p-1}(1-x)^{q-1}$, linked to the BeTaBoOp$(p,q,1,1) \equiv$ Beta(p,q) density, is:

   – Proportional to the product of the densities of $U_{p:p}$ and $U_{1:q}$:

     Since $U_{1:q} \preceq U_{p:p}$, for all $p,q \in \mathbb{N}$, and $U_{p:p}$ is associated with the growth term x^{p-1}, population growth is observed. However, if $p=q$, the retroaction term ruled by $U_{1:p}$ will curb down the population growth to sustainable levels, since $U_{1:p}$ and $U_{p:p}$ are equally distant order statistics from the extremes, in the sense that they are of the type $U_{k:n}$ and $U_{n-k+1:n}$. Therefore, when $p=q$, we may think that $U_{1:p}$ and $U_{p:p}$ are exerting equal opposite effects, and thus ensuring a sustainable growth. On the other hand, if $p \neq q$, uncontrolled population dynamics is the case.

   – Or proportional to the density of $U_{p:p+q-1}$:

     If $p=q$, then $U_{p:2p-1}$ is the median of $2p-1$ independent standard uniform random variables, hence reinforcing the idea of sustainable growth, i.e. population equilibrium, as seen above. But if $p \neq q$, we are dealing with uncontrolled population dynamics, since $U_{p:p+q-1} \preceq U_{\lfloor (p+q-1)/2 \rfloor +1:p+q-1}$ for $p<q$, and $U_{p:p+q-1} \succeq U_{\lfloor (p+q-1)/2 \rfloor +1:p+q-1}$ for $p>q$, where $U_{\lfloor (p+q-1)/2 \rfloor +1:p+q-1}$ is the median of $p+q-1$ independent standard uniform random variables.

2. The expression $x^{p-1}(-\ln x)^{Q-1}$, linked to the BeTaBoOp$(p,1,1,Q) \equiv$ Betinha(p,Q) density, is:

   – Proportional to the product of the densities of $U_{p:p}$ and $U_Q^{(*)}$:

     From the fact that $U_Q^{(*)} \preceq U_{p:p}$, for all $p, Q \in \mathbb{N}$, the growth term is again the dominant one, and consequently population growth is also observed in this setting. Now the question is whether it is possible to have here sustainable growth. The answer is no, because if we compare the two retroaction terms $(1-x)^{Q-1}$ and $(-\ln x)^{Q-1}$, which are proportional to the densities of $U_{1:Q}$ and $U_Q^{(*)}$, respectively, we have $U_Q^{(*)} \preceq U_{1:Q}$. Therefore, $U_Q^{(*)}$ exerts a weaker control effect on population growth than $U_{1:Q}$, which leads necessarily to unsustained population growth, even if $Q = p$.

   – Or proportional to the density of $U_Q^{1/p(*)}$, which applies to the more general case $p > 0$:

     Here, $U_Q^{1/p(*)} \preceq U_Q^{(*)}$ if $p > 1$, and $U_Q^{(*)} \preceq U_Q^{1/p(*)}$ if $p < 1$. By comparing $U_Q^{1/p(*)}$ and $U_Q^{(*)}$ with $U_{1:Q}$, which is associated with the retroaction factor $(1-x)^{Q-1}$, we conclude that:

     (i) for $p > 1$, $U_Q^{(*)} \preceq U_{1:Q}$, thus revealing that $U_Q^{1/p(*)}$ has a weaker control effect on population growth, as already unveiled above;

     (ii) for $p < 1$, $U_{1:Q} \preceq U_Q^{1/p(*)}$, therefore showing that $U_Q^{1/p(*)}$ has a stronger control effect on population growth.

     Both cases are suitable to model unsustainable population growth.

3. The expression $(1-x)^{q-1}(-\ln(1-x))^{P-1}$, tied to the BeTaBoOp$(1,q,P,1)$ density, is proportional to the product of the densities of $U_{1:q}$ and $1-U_P^{(*)}$, associated with the retroaction and growth terms $(1-x)^{q-1}$ and $(-\ln(1-x))^{P-1}$, respectively.

   Since $U_{1:q} \preceq 1 - U_P^{(*)}$ for all $q, P \in \mathbb{N}$, the growth factor is the dominant one, and therefore population growth will also happen. On the other hand, from the fact that $U_{P:P} \preceq 1 - U_P^{(*)}$, where $U_{P:P}$ is associated with the (absent) growth term x^{P-1}, shows that in this case we have a stronger growth impetus, counteracted by growth control mechanisms influenced by $U_{1:q}$. As $U_{1:q}$ exerts a stronger control effect than $U_q^{(*)}$ would on population growth, this case is also suitable for modeling populations with unsustainable growth, as the previous one, but where a more uncontrolled population growth is observed.

   Recall that Brilhante *et al.* [2] showed that the normalized solution for the differential equation linked to the BeTaBoOp$(1,2,2+\alpha,1)$ density belongs to the class of extreme value laws for minima, which seems to be the consequence of the higher control forces needed to refrain a more uncontrolled population growth through the influence of $U_{1:q}$.

4. The expression $x^{p-1}(-\ln(1-x))^{P-1}$, tied to the BeTaBoOp$(p,1,P,1)$ density, is proportional to the product of the densities of $U_{p:p}$ and $1-U_P^{(*)}$, with $U_{p:p} \preceq 1-U_P^{(*)}$ only if $p \leq P$. Thus, the growth pattern which is linked with the factor x^{p-1} is the dominant one, whenever $p \leq P$.

   Because the growth control mechanisms are absent in this setting, the associated differential equation is ideal for modeling populations that almost surely grows to infinity, extinction being almost impossible.

5. The expression $(1-x)^{q-1}(-\ln x)^{Q-1}$, linked to the BeTaBoOp$(1,q,1,Q)$ density, is proportional to the product of densities of $U_{1:q}$ and $U_Q^{(*)}$, where $U_Q^{(*)} \preceq U_{1:q}$ if $q \leq Q$. Therefore, the retroaction term tied to $(1-x)^{q-1}$ is the dominant one, whenever $q \leq Q$.

   Given that we only have growth control factors here, the corresponding differential equation is useful for modeling populations that are almost surely doomed to extinction.

6. The expression $(-\ln(1-x))^{P-1}(-\ln x)^{Q-1}$, linked to the BeTaBoOp$(1,1,P,Q)$ density, is proportional to the product of densities of $1-U_P^{(*)}$ and $U_Q^{(*)}$, with $U_Q^{(*)} \preceq 1-U_P^{(*)}$ for all $P,Q \in \mathbb{N}$. In this setting population growth is observed, with sustainable growth occurring whenever the growth parameter P and the retroaction parameter Q are equal.

7. The expression $x^{p-1}(1-x)^{q-1}(-\ln x)^{Q-1}$, tied to the BeTaBoOp$(p,q,1,Q)$ density, is proportional to the product of the densities of $U_{p:p}$, $U_{1:q}$ and $U_Q^{(*)}$, with $U_Q^{(*)} \preceq U_{1:q} \preceq U_{p:p}$ if $q \leq Q$. Again population growth is noticed since the dominant term is the growth term.

   However, when $p=q=Q$, $U_{1:p}$ manages to "compensate" the growth effect of $U_{p:p}$ by curbing down the population growth to sustainable levels. This action is reinforced by the other retroaction term $(-\ln x)^{p-1}$ ruled by $U_p^{(*)}$. A more interesting case occurs when the growing parameter p and the retroaction parameters q and Q meet an equilibrium, in the sense that $p=q+Q$.

8. The expression $x^{p-1}(1-x)^{q-1}(-\ln(1-x))^{P-1}$, linked to the BeTaBoOp$(p,q,P,1)$ density, is proportional to the product of the densities of $U_{p:p}$, $U_{1:q}$ and $1-U_P^{(*)}$, with $U_{1:q} \preceq U_{p:p} \preceq 1-U_P^{(*)}$ for $p \leq P$.

   Uncontrolled population growth is again the case here even if $p=q=P$. This is so because although $U_{1:p}$ "compensates" the effect of $U_{p:p}$, it does not do the same for the growth term ruled by $1-U_p^{(*)}$, whose influence is stronger than $U_{p:p}$. However, an equilibrium is observed when the growing parameters p and P and the retroaction parameter q verify the relation $p+P=q$.

9. The expression $x^{p-1}(-\ln(1-x))^{P-1}(-\ln x)^{Q-1}$, linked to the BeTaBoOp$(p,1,P,Q)$ density, is proportional to the product of the densities of $U_{p:p}$, $1-U_P^{(*)}$ and $U_Q^{(*)}$, with $U_Q^{(*)} \preceq U_{p:p} \preceq 1-U_P^{(*)}$. In this

setting we shall have uncontrolled population growth, unless equilibrium is meet, i.e. when $p + P = Q$.

10. The expression $(1 - x)^{q-1}(-\ln(1 - x))^{P-1}(-\ln x)^{Q-1}$, tied to the BeTaBoOp$(1, q, P, Q)$ density, is proportional to the product of the densities of $U_{1:q}$, $1 - U_P^{(*)}$ and $U_Q^{(*)}$, where $U_Q^{(*)} \preceq U_{1:q} \preceq 1 - U_P^{(*)}$, if $q \leq Q$. Here we have two control mechanisms acting on population growth, with sustainability being achieved if $P = q + Q$.
11. The expression $x^{p-1}(1 - x)^{q-1}(-\ln(1 - x))^{P-1}(-\ln x)^{Q-1}$, linked to the BeTaBoOp$(p, q, P, Q)$ density, is proportional to the product of the densities of $U_{p:p}$, $U_{1:q}$, $1 - U_P^{(*)}$ and $U_Q^{(*)}$, where $U_Q^{(*)} \preceq U_{1:q} \preceq U_{p:p} \preceq 1 - U_P^{(*)}$ if $p \leq P$ and $q \leq Q$.

    In this setting equilibrium is observed when $p + P = q + Q$.

From the exposed above, we see that the generalized Verhulst theory is quite versatile in describing a wide range of population dynamics.

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