Dynamic Instabilities in Population Growth Models II: Panjer Randomized Modified Fibonacci Model

Maria de Fátima Brilhante[1,4], Maria Ivette Gomes[2,4], Dinis Pestana[3,4], and Maria Luísa Rocha[4,5,6]

[1] Universidade dos Açores, Departamento de Matemática, Ponta Delgada, Açores, Portugal
(E-mail: fbrilhante@uac.pt)
[2] Universidade de Lisboa, Faculdade de Ciências, DEIO, Portugal, and Instituto de Investigação Científica Bento da Rocha Cabral, Lisboa, Portugal
(E-mail: ivette.gomes@fc.ul.pt)
[3] Universidade de Lisboa, Faculdade de Ciências, DEIO, Portugal, and Instituto de Investigação Científica Bento da Rocha Cabral, Lisboa, Portugal
(E-mail: dinis.pestana@fc.ul.pt)
[4] CEAUL — Centro de Estatística e Aplicações da Universidade de Lisboa, Portugal
[5] Universidade dos Açores, Departamento de Economia e Gestão, Ponta Delgada, Açores, Portugal
(E-mail: lrocha@uac.pt)
[6] CEEAplA — Centro de Estudos de Economia Aplicada do Atlântico

Abstract. Branching processes are natural models for random population growth in many situations. Here we use basic count models whose probability mass function satisfies Panjer iteration, and investigate randomly stopped sums and collective risk when the subordinator random variable and the summands are independent and identically distributed basic count random variables.
Keywords: Branching processes, Panjer iteration, basic count models, collective reisk, fixed point algorithm instabilities.

1 Randomizing the Fibonaccy Population Growth Model Via Branching Processes

Fibonacci (c. 1170 – c. 1250) in his *Liber Abaci* posed and solved a problem involving the growth of a population of rabbits based on idealized and very unrealistic assumptions. As a consequence, a population with Fibonacci's growth pattern never dies out, while we know that the total progeny of some ancestor is in many real circumstances finite, cf. for instance Lotka [10] example (p. 123–136) on the extinction of surnames, using branching processes.

Let $\{f_n\}_{n\in\mathcal{S}_X}$ denote the probability mass function (pmf) of a discrete random variable (rv) X with support $\mathcal{S}_X \subset \mathbb{N}$. The corresponding probability generating function (pgf) is $m_X(t) = E(t^X) = \sum_{n=0}^{\infty} f_n t^n$.

If N is a discrete rv, $X_0 = 0$ and $X_1, X_2, ...$ independent replicas of X, with N and X_k independent, and we define the "compound" rv $Y = \sum_{k=0}^{N} X_k$, then

$$m_Y(t) = \sum_{j \in \mathcal{S}_Y} m_X^j(t)\, \mathbb{P}[N = j] = m_N(m_X(t)).$$

From this, we may easily compute mean value and variance of the rv Y. An alternative designation for the concept of compounding rv's is the concept of randomly stopped sums, which can have the advantage of explicitly indicating the type of the subordinator rv.

If in particular $X_k,\ k = 1, 2, ...$ are independent replicas of a count rv X modeling the number of direct descendants of each individual (or each female) in the population, and we define

$$Y_0 = 1, \qquad Y_1 = X_1, \qquad Y_2 = \sum_{k=0}^{Y_1} X_k, \quad ... \quad Y_{n+1} = \sum_{k=0}^{Y_n} X_k, \quad ...$$

we may interpret Y_k as the number of direct offsprings in the k-th generation, and $Z_n = \sum_{j=0}^{n} Y_j$ as the total progeny of some ancestor until the n-th generation. Let us denote $m(t) = m_1(t)$ the pgf of $Y_1 \stackrel{\mathrm{d}}{=} X$, $m_n(t)$ the pgf of Y_n; then $m_n(t) = m(m_{n-1}(t) = m^{\otimes(n)}(t)$, where $m^{\otimes(n)}$ denotes the n-fold composition of m with itself.

Following Good [5] (an argument that inspired Feller [3], XII.5), $m_{Z_1}(t) = t\, m_X(t)$ and iteratively $m_{Z_n}(t) = t\, m_{Z_{n-1}}(t)$, we obtain the probability generating functions for the number of descendants up to each successive generation.

This is a decreasing sequence, whose limit $\rho(s)$ satisfies $\rho(s) = s\, m_X(\rho(s))$ and which may be found solving $t = s\, m_X(t)$. Each coefficient r_k in the MacLaurin's expansion of $\rho(s)$ is the probability that the total progeny consists of k elements, and therefore if $\sum r_k = \rho(1) < 1$, this is the probability of extinction.

$\{Y_0, Y_1, ...\}$ is usually called a Galton–Watson branching process, or a cascade process. Simple examples of branching processes, and basic results on important problems such as extinction probability and size of a population can be found in Feller [3]. Namely, in what concerns extinction:

Theorem 1. *If $\mathbb{E}(Y) = \mu \leq 1$, the process almost surely dies out, and its expected size is $\dfrac{1}{1-\mu}$ when $\mu < 1$, and infinite when $\mu = 1$. If $\mu > 1$, the probability f_n that the process terminates at or before the n-th generation tends to the unique root $x < 1$ of the equation $x = m_Y(x)$.*

And, in what concerns the total progeny:

Theorem 2. *Denoting ρ_k the probability that the total progeny has k individuals,*

1. *the extinction probability is* $\sum_{k=1}^{\infty} \rho_k$.
2. *The pgf* $\rho(s) = \sum_{k=1}^{\infty} \rho_k s^k$ *is given by the unique positive root of* $t = s\, m_Y(t)$, *and* $\rho(s) \leq x$.

More extensive monographies on branching processes, with deeper results, are Harris [6], Athreya and Ney [1] or Jaegers [8]. Gnedenko and Korolev [4] present interesting examples of random infinite divisibility and random stability using branching processes, and they establish necessary and sufficient conditions for the convergence of randomly stopped sums, and limit theorems for super-critical (i. e., $\mu = \mathbb{E}(X) > 1$) Galton–Watson processes.

In [2], Brilhante *et al.* investigated randomization of the Fibonacci's growth pattern modeling the individual progeny at a mating epoch using *Bernoulli*(p), and thus the progeny of the initial ancestor as

$$Z_1 = \begin{cases} 0 & 1 & 2 & 3 \\ (1-p)^2 & p(1-p)(2-p) & 2p^2(1-p) & p^3 \end{cases}$$

(since only two mating epochs are permitted to each individual).

The $Y \frown$ *Geometric*(p) model for the number of direct descendants, with pmf $\{f_n = p\,(1-p)^n\}_{n\in\mathbb{N}}$, provides an algebraic simple treatment. In fact, writing $q = 1-p$, $m_Y(t) = \dfrac{p}{1-qt}$, and

$$m_{Y_n}(t) = \begin{cases} p\,\dfrac{q^n - p^n - (q^{n-1} - p^{n-1})\,q\,t}{q^{-1}n - p^{n-1} - (q^n - p^n)\,q\,t} & p \neq q \\[2ex] \dfrac{n-(n-1)t}{n+1-nt} & p = q = \frac{1}{2} \end{cases}$$

is easily computed.

Both the *Bernoulli*(p) and the *Geometric*(p) pmf's satisfy the recursive expression

$$f_{n+1} = \left(a + \frac{b}{n+1}\right) f_n, \ \forall n \geq k, \quad f_n = 0 \text{ for } 0 \leq n \leq k-1$$

(in the case of $X \frown$ *Bernoulli*(p), $a = \dfrac{p}{p-1}$ and $b = \dfrac{2p}{1-p}$, and in the case of $X \frown$ *Geometric*(p), $a = q$ and $b = 0$). As we shall state in the following section, the above recursive expression is valid for the pmf of a broad class of rv's, known as Panjer rv's, that play an important role on the theory of collective risk. We investigate some consequences of using simple Panjer direct progeny models in branching processes.

2 Basic Count Models

We shall say that X is a Panjer rv if its pmf $\{f_n\}_{n\in\mathcal{S}_X}$ satisfies the recursive expression

$$f_{n+1} = \left(a + \frac{b}{n+1}\right) f_n, \ \forall n \geq k, \quad f_n = 0 \text{ for } 0 \leq n \leq k-1. \tag{1}$$

We denote $Panjer(a, b, k)$ the class of all pmf's satisfying (1).

This expression has been used by several authors, with $k = 0$, before Panjer [11], but it was in this seminal paper that the consequences for the iterative computation of the density of the collective risk process have been established.

In fact, Panjer [11] considered only the case $k = 0$ — for which the non degenerate types are the underdispersed binomial, the overdispersed negative binomial, and the Poisson in between —, but immediatly Sundt and Jewell [14] published the extension for $k = 1$, with the logarithmic and the extended negative binomial solutions.

Finally Hess *et al.* [7] defined the general class, with the recursion starting with $k \geq 0$, the $f_0, ..., f_{k-1}$ being free parameters (for $k = 0$, f_0 can be considered the starting jump of a hurdle process); it is also known as the class of basic count distributions, or class of basic claim distributions. For more details, cf. Rolsky *et al.* [13], Klugman *et al.* [9], and Pestana and Velosa [12].

Theorem 3. *Let $\{f_n\}_{n\in\mathcal{S}_X}$ be the pmf of a non degenerate count rv X. For a, $b \in \mathbb{R}$ the statements that follow are equivalent:*

(a) $\{f_n\}_{n\in\mathcal{S}_X}$ is a $Panjer(a, b; k)$ pmf.
(b) for $\ell \in \mathbb{N}^+$, the pgf $m_X(t) = \sum_{n=0}^{\infty} f_n t^n$ satisfies the differential equations

$$(1 - at)h^{(\ell)}(t) = (\ell a + b)h^{(\ell-1)}(t) + f_k \binom{k}{\ell} \ell!\, t^{k-1},$$

$t \in [0, 1)$ and $h^{(j)}(0) = 0$ for $j \leq k - 1$.
(c) m_X satisfies the differential equation

$$(1 - at)h^{(k+1)}(t) = ((k + 1)a + b)h^{(k)}(t),$$

$t \in [0, 1)$ and $h^{(j)}(0) = 0$ for $j \leq k - 1$.

Further, $Q = Panjer(a, b; k) \implies (k + 1)\, a + b > 0$, and on the other hand $a + b \geq 0 \Rightarrow a < 1$ and $a + b < 0 \Rightarrow a \leq 1$.

From this it is easy to conclude that the Panjer class has the following non degenerate elements:

1. The $Binomial(\nu, p),\ \nu \in \mathbb{N}^+,\ p \in (0,1)$, which is $Panjer(\frac{p}{p-1}, \frac{(\nu+1)\,p}{1-p}, 0)$. Its variation index $\mathcal{I}(X) = \dfrac{\mathrm{var}(X)}{\mathbb{E}(X)} = 1 - p < 1$, i.e., X is underdispersed.
2. The $Poisson(\mu),\ \mu > 0$ is $Panjer(0, \mu, 0)$. Its dispersion index is 1.
3. The overdispersed $NegativeBinomial(\alpha, p),\ \alpha > 0,\ p \in (0,1)$, with pmf $\left\{\binom{\alpha+n-1}{n} p^n (1-p)^\alpha\right\}_{n\in\mathbb{N}}$, is $Panjer(p, (\alpha-1)\,p, 0)$.
4. The $ExtendedNegativeBinomial(\alpha, p, k),\ \alpha \in (-k, -k+1),\ p \in (0,1), k \in \mathbb{N}^+$, with pmf
$$f_n = \frac{\dbinom{\alpha+n-1}{n} p^n}{(1-p)^{-\alpha} - \displaystyle\sum_{j=0}^{k-1} \dbinom{\alpha+j-1}{j} p^j}, \qquad n = k, k+1, ...,$$
in the support $\mathcal{S}_X = \{k, k+1, \dots\}$, is $Panjer(p, (\alpha-1)\,p, k)$. In the expression above the extended binomial coefficients $\dbinom{\alpha+n-1}{n}$ are defined as $\dbinom{\alpha+n-1}{n} = \dbinom{-\alpha}{n} = \dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)\,n!}$.
5. The $ExtendedLogarithmic(p, k),\ p \in (0,1), k \in \mathbb{N}^+$, with pmf
$$f_n = \frac{1}{\displaystyle\sum_{j=m}^{\infty} \frac{p^j}{\dbinom{j}{m}}} \frac{p^n}{\dbinom{n}{m}}, \qquad n = k, k+1, ...,$$
is $Panjer(p, -kp, k)$.
6. If $X \frown Panjer(a, b, k)$, truncating $\{k, k+1, ..., \ell - 1\} \subset \mathcal{S}_X$ we obtain a truncated rv $X^* \frown Panjer(a, b, \ell)$.

The special "unit" cases $Bernoulli(p) \equiv Binomial(1, p)$, $Geometric(p) \equiv NegativeBinomial(1, p))$, $ExtendedNegativeBinomial(\alpha, p, 1)$ whose pmf has the simple form $\dfrac{1-(1-q\,t)^{-\alpha}}{1-p^{-\alpha}},\ t \le \frac{1}{q})$, and $Panjer(p, -p, 1)$ or $Logarithmic(p)$ (or $ExtendedLogarithmic(1, p)$, with pgf $\dfrac{\ln(1-pt)}{\ln(1-p)}$), do have specially nice properties in each of the corresponding subclasses.

In particular, $NegativeBinomial(\alpha, p)$ — and hence, as a special case $Geometric(p)$ — that result from a *Gamma* randomization of the $Poisson(\Lambda)$, i.e., an hierarchic model with $\Lambda \frown Gamma(\alpha, 1)$ — are successfully used to model the descendance of populations when the distribution of direct offsprings exhibits large variation, and both the the $ExtendedNegativeBinomial(\alpha, p, 1)$ and the $Logarithmic(p)$ distributions have been used to provide close fit to some natural populations.

In Table 1 below we sumarize results, indicating also the pgf:

Table 1. Panjer distributions.

X	a	b	k	$m_Q(t)$
$Binomial(m,p)$	$\frac{p}{p-1}$	$\frac{(m+1)p}{1-p}$	0	$(1-p+pt)^m$
$Poisson(\mu)$	0	μ	0	$e^{\mu(t-1)}$
$NegativeBinomial(\alpha,p)$	p	$(\alpha-1)p$	0	$(\frac{1-pt}{1-p})^{-\alpha}$
$ExtendedNegativeBinomial(\alpha,p,k)$	p	$(\alpha-1)p$	k	$\frac{(1-pt)^{-\alpha}-\sum_{j=0}^{k-1}\binom{\alpha+j-1}{j}(pt)^j}{(1-p)^{-\alpha}-\sum_{j=0}^{k-1}\binom{\alpha+j-1}{j}p^j}$
$ExtendedLogarithmic(p,k)$	p	$-kp$	k	$\frac{\sum_{n=k}^{\infty}\binom{n}{k}^{-1}(pt)^n}{\sum_{n=k}^{\infty}\binom{n}{k}^{-1}p^n}$

3 Randomly Stopped Sums with Panjer Subordinator

The importance of the Panjer class is a consequence of the implications that the recursive expression (1) has on the recursive computation of the density of randomly stopped sums subordinated by Panjer rv's. This results from the following theorem:

Theorem 4. *Let $\{q_n\}_{n\in\mathbb{N}}$ be the pmf of a count distribution Y, and $\{f_n\}_{n\in\mathbb{N}}$ denote the pmf of a claim number distribution X whose support is a subset of the positive integers, i. e. $f_0 = 0$. Consider the randomly stopped sum $T = \sum_{n=\inf \mathcal{S}_Y}^{Y} X_n$, with Y and the replicas X_n of X independent.*

Then the following statements are equivalent:

1. *$Y \frown Panjer(a,b,k)$;*
2. *For any claim number rv X and any $\ell \geq 1$, m_T satisfies the differential equation*

$$(1-a\,m_X(t))h^{(\ell)}(t) = \sum_{i=1}^{\ell}\binom{\ell}{i}(a+b\frac{i}{\ell})\,h^{(\ell-i)}(t)\,m_X^{(i)}(t) + q_k\,m_T^{(\ell)}(t),$$

$t \in [0,1)$, with the initial conditions $h^{(j)}(0) = 0$ for $j \leq k-1$.

From this, we can compute the pmf of a compound rv T with Panjer subordinator Y and count summands independent replicas of X, as defined above, by observing that for $\ell \geq 1$

$$(1 - a\, m_X(t))\; m_T^{(\ell)}(t) = \sum_{i=1}^{\ell} \binom{\ell}{i} \left(a + b\,\frac{i}{\ell}\right) m_T^{(\ell-i)}(t)\, m_X^{(i)}(t) + q_k\,[m_X^k(t)]^{(\ell)}.$$

In fact, the main consequence of Panjer's theory is the following result:

Theorem 5. *Let $\{q_n\}_{n\in\mathbb{N}}$ be the pmf of a count distribution Y, and $\{f_n\}_{n\in\mathbb{N}}$ denote the pmf of a claim number distribution X whose support is a subset of the positive integers. Consider the randomly stopped sum $T = \sum_{n\in\mathcal{S}_Y} X_n$, with Y and the replicas X_n of X independent. Then*

$$\mathbb{P}[T = n] = g_n = \begin{cases} m_Y(m_X(0)) = m_T(f_0) & n = 0 \\[2ex] \dfrac{1}{1 - a\, f_0}\left[\displaystyle\sum_{i=1}^{n} \left(a + b\frac{i}{n}\right) g_{n-i} f_i\right] + q_k f_n^{*k} & n \geq 1 \end{cases}$$

where f_n^{*k} stands for the k-th iterated convolution of the sequence $\{f_n\}$ with itself.

(There exists a simple extension for the density when the summands are absolutely continuous, but it is not relevant in the context of branching processes.)

4 Discussion and Conclusions

With the exception of *Poisson* or of *Geometric* subordinator — i.e., of a $Panjer(0, \mu, 0)$ or a $Panjer(p, 0, 0)$, respectively, cf. Pestana and Velosa [12] on the simplicity of these cases when compared to the complexity of others — we couldn't obtain any close expressions for the n-fold composition of the pgf for any other Panjer subordinators. Aside from those two cases, the only one for which we got more promising results has been — as predictable — the *Logarithmic*(p). Moreover, when the aim is to extend the Fibonacci sequence using branching randomization, in case we want to remove individuals from the population after two mating epochs, we have the extra burden of subtracting, the two rv's used being dependent.

Happily, compound pgf's are amenable to compute mean values and variances, and in what concerns the mean value we have the extra facility that the mean value of the difference is the difference of the means values, regardless whether the random variables are dependent or independent. So, it is easy to follow the process on average, and the relation of the sequence of expected values to the sequence of Fibonacci numbers simple.

The quantities of interest — extinction probability and expected total size in the supercritical case, size of the n-th generation, total size of the population up to the n-th generation — can be dealt with computationally. When the fixed point method is used to compute roots of some equation $F(x) = x$, numerical instabilities are a rule whenever F is too steep, and the sufficient convergence conditions are not met.

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