Extensions of Verhults Model in Population Dynamics and Extremes

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Abstract. Starting from the Beta(2,2) model, connected to the Verhulst logistic parabola, several extensions are discussed, and connections to extremal models are revealed.
Aside from the classical GEV (General Extreme Value Model) from the iid case, extreme value models in randomly stopped extremes schemes are discussed; in this context, the classical logistic Verhulst model is a max-geo-stable model, i.e. geometric thinning of the observations curbs down growth to sustainable patterns. The general differential models presented are a unified approach to population dynamics growth, with factors of the form $[-\ln(1-N(t))]^{P-1}$ and the linearization $[N(t)]^{p-1}$ modeling two very different growth patterns, and factors of the form $[-\ln N(t)]^{Q-1}$ and the linearization $[1-N(t)]^{q-1}$ leading to very different ambiental resources control of the growth behavior.
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1 Introduction

Let $N(t)$ denote the size of some population at time t. Verhults [16], [17], [18] imposed some natural regularity conditions on $N(t)$, namely that $\frac{\mathrm{d}}{\mathrm{d}t}N(t) = \sum_{k=0}^{\infty} A_k[N(t)]^k$, with $A_0 = 0$ since nothing can stem out from an extinct population, $A_1 > 0$ a "growing" parameter, $A_2 < 0$ a retroaction parameter controlling sustainable growth tied to available resources, see also Lotka [9].

The second order approximation $\frac{\mathrm{d}}{\mathrm{d}t}N(t) = A_1 N(t) + A_2[N(t)]^2$ can be rewritten

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)\left[1 - \frac{N(t)}{K}\right] \tag{1}$$

where $r > 0$ is frequently interpreted as a Malthusian instantaneous growth rate parameter when modeling natural breeding populations, and $K > 0$ as the equilibrium limit size of the population.

The general form of the solution of the (1) differential equation approximation is the family of logistic functions $N(t) = \dfrac{K\,N_0}{N_0 + (K - N_0)\,\mathrm{e}^{-rt}}$ (where N_0 is the population size at time $t = 0$), and this is the reason why in the context of population dynamics $r\,x\,(1-x)$ is frequently referred to as "the logistic parabola".

Due to the seasonal reproduction and time life of many natural populations, the differential equation (1) is often discretized, first taking r^* such that $N(t+1) - N(t) = r^*\,N(t)\left[1 - \frac{N(t)}{K}\right]$ and then $\alpha = r^* + 1$, $x(t) = \frac{r^*\,N(t)}{r^*+1}$, to obtain $x(t+1) = \alpha\,x(t)[1 - x(t)]$, and then the asociated difference equation

$$x_{n+1} = \alpha x_n\,[1 - x_n], \tag{2}$$

where it is convenient to deal with the assumption $x_k \in [0,1],\ k = 1,2,\ldots$ The equilibrium $x_{n+1} = x_n$ leads to a simple second order algebraic equation with positive root $1 - \frac{1}{\alpha}$, and to a certain extent it is surprising that anyone would care to investigate its numerical solution using the fixed point method, which indeed brings in many pathologies when a steep curve — i.e., for some values of the iterates $|\alpha\,(1 - 2\,x_k)| > 1$ — is approximated by an horizontal straight line. This numerical investigation, apparently devoid of interest, has however been at the root of many theoretical advances (namely Feigenbaum bifurcations and ultimate chaotic behavior), and *a posteriori* led to many interesting breakthroughs in the understanding of population dynamics.

Observe also that (2) may be rewritten $x_{n+1} = \frac{\alpha}{6}\,6\,x_n\,[1 - x_n]$, and that $f(x) = 6\,x\,(1-x)\,\mathrm{I}_{(0,1)}(x)$ is the $Beta(2,2)$ probability density function. Extensions of the Verhults model using difference equations similar to (2), but where the right hand side is tied to a more general $Beta(p,q)$ probability density function have been investigated in Aleixo *et al.* [1] and in Rocha *et al.* [13].

Herein we consider further extensions of population dynamics first discussed in Pestana *et al.* [10], Brilhante *et al.* [5] and in Brilhante *et al.* [3], whose inspiration has been to remark that $1-x$ is the linear truncation of the series expansion of $-\ln x$, as well as x is the linear truncation of the series expansion of $-\ln(1-x)$.

In Section 2, we describe the $BeTaBoOp(p,q,P,Q),\ p,q,P,Q > 0$ family of probability density functions, with special focus on subfamilies for which one at least of those shape parameters is 1.

In section 3, some points tying population dynamics and statistical extreme value models are discussed, namely discussing the connection of the instantaneous growing factors x^{p-1} and $[-\ln(1-x)]^{P-1}$ to models for minima, and the retroaction control factors $(1-x)^{q-1}$ and $[-\ln x]^{Q-1}$ to modeling population growth using maxima extreme value models — either in the classical extreme value setting, either in the geo-stable setting, where the geometric thinning curbs down growth to sustainable patterns.

2 The $X_{p,q,P,Q} \frown BeTaBoOp(p,q,P,Q)$ models, $p,q,P,Q>0$

Let $\{U_1,U_2,\ldots,U_Q\}$ be independent and identically distributed (iid) standard uniform random variables, $V=\prod_{k=1}^{Q} U_k^{\frac{1}{p}}$, $p>0$ the product of iid $Beta(p,1)$ random variables. As $-\ln V \frown Gamma(Q,\frac{1}{p})$, the probability density function (pdf) of V is $f_V(x)=\frac{p^Q}{\Gamma(Q)}\,x^{p-1}(-\ln x)^{Q-1}\mathrm{I}_{(0,1)}(x)$.

Brilhante *et al.* [5] discussed the more general $Betinha(p,Q)$ family of random variables $\{X_{p,Q}\}$, $p,Q>0$, with pdf

$$f_{X_{p,Q}}(x)=\frac{p^Q}{\Gamma(Q)}\,x^{p-1}(-\ln x)^{Q-1}\mathrm{I}_{(0,1)}(x),\ p,Q>0$$

that can be considered an extension of the $Beta(p,q)$, $p,q>0$ family, since $1-x$ is the linearization of the MacLaurin expansion $-\ln x=\sum_{k=1}^{\infty}\frac{(1-x)^k}{k}$ to derive population growth models that do not comply with the sustainable equilibrium exhibited by the Verhulst logistic growth model.

On the other hand, if $X_{q,P} \frown Betinha(q,P)$, the pdf of $1-X_{q,P}$ is

$$f_{1-X_{q,P}}(x)=\frac{q^P}{\Gamma(P)}\,(1-x)^{q-1}(-\ln(1-x))^{P-1}\mathrm{I}_{(0,1)}(x),\ q,P>0,$$

and the family of such random variables also extends the $Beta(p,q)$ family in the sense that x is the linearization of $-\ln(1-x)$.

Having in mind Hölder's inequality, it follows that

$$x^{p-1}(1-x)^{q-1}[-\ln(1-x)]^{P-1}(-\ln x)^{Q-1}\in\mathcal{L}^1_{(0,1)},\ p,q,P,Q>0,$$

and hence

$$f_{X_{p,q,P,Q}}(x)=\frac{x^{p-1}(1-x)^{q-1}[-\ln(1-x)]^{P-1}(-\ln x)^{Q-1}\mathrm{I}_{(0,1)}(x)}{\displaystyle\int_0^1 x^{p-1}(1-x)^{q-1}[-\ln(1-x)]^{P-1}(-\ln x)^{Q-1}\mathrm{d}x} \qquad (3)$$

is a pdf for all $p,q,P,Q>0$. Obviously, $1-X_{p,q,P,Q}=X_{q,p,Q,P}$. For simplicity, in what follows we shall use the lighter notation $f_{p,q,P,Q}$ instead of $f_{X_{p,q,P,Q}}$ for the density of $X_{p,q,P,Q}$.

Brilhante *et al.* [3] used the notation $X_{p,q,P,Q} \frown BeTaBoOp(p,q,P,Q)$ for the random variable with pdf (3) — obviously the $Beta(p,q)$, $p,q>0$

family of random variables is the subfamily $BeTaBoOp(p, q, 1, 1)$, and the formerly introduced $Betinha(p, Q)$, $p, Q > 0$ is in this more general setting the $BeTaBoOp(p, 1, 1, Q)$ family. The cases for which some of the shape parameters are 1 and the other parameters are 2 are particularly relevant in population dynamics. In the present paper, we shall discuss in more depth $X_{p,1,1,Q}$ and $X_{1,q,P,1}$, and in particular $X_{2,1,1,2}$ and $X_{1,2,2,1}$.

Some of the 15 subfamilies when one or more of the 4 shape parameters p, q, P, Q are 1 have important applications in modeling; below we enumerate the most relevant cases, giving interpretations, for integer parameters, in terms of products of powers of independent $U_k \frown Uniform(0, 1)$ random variables.

1. $X_{1,1,1,1} = U \frown Uniform(0, 1)$; $f_{1,1,1,1}(x) = \mathrm{I}_{(0,1)}(x)$.
2. $X_{p,1,1,1} = U^{\frac{1}{p}} \frown Beta(p, 1)$; $f_{p,1,1,1}(x) = p\, x^{p-1}\mathrm{I}_{(0,1)}(x)$.
3. $X_{1,q,1,1} = 1 - U^{\frac{1}{q}} \frown Beta(1, q)$; $f_{1,q,1,1}(x) = q\,(1 - x)^{q-1}\mathrm{I}_{(0,1)}(x)$.
4. $X_{1,1,P,1}$, that for $P \in \mathbb{N}$ is 1 minus the product of P iid standard uniform random variables,

   $$X_{1,1,P,1} = 1 - \prod_{k=1}^{P} U_k, \quad U_k \frown Uniform(0, 1), \text{independent.}$$

   More generally, for all $P > 0$, $f_{1,1,P,1}(x) = \dfrac{(-\ln(1 - x))^{P-1}}{\Gamma(P)}\, \mathrm{I}_{(0,1)}(x)$, where $\Gamma(P) = \displaystyle\int_0^\infty x^{P-1}\mathrm{e}^{-x}\mathrm{d}x$ is Euler's gamma function.
5. $X_{1,1,1,Q}$, that for $Q \in \mathbb{N}$ is the product of P iid standard uniform random variables,

   $$X_{1,1,1,Q} = \prod_{k=1}^{Q} U_k, \quad U_k \frown Uniform(0, 1), \text{independent;}$$

   alternatively, $X_{1,1,1,Q}$ may be described in the following hierarchical construction: denote $Y_1 \overset{\mathrm{d}}{=} X_{1,1,1,1} \frown Uniform(0, 1)$, $Y_2 \frown Uniform(0, Y_1)$, $Y_3 \frown Uniform(0, Y_2)$, $\ldots$, $Y_Q \frown Uniform(0, Y_{Q-1})$. Then $Y_Q \overset{\mathrm{d}}{=} X_{1,1,1,Q} \frown BeTaBoOp(1, 1, 1, Q)$.

   More generally, for all $Q > 0$, $f_{1,1,1,Q}(x) = \dfrac{(-\ln(x))^{Q-1}}{\Gamma(Q)}\, \mathrm{I}_{(0,1)}(x)$.
6. $X_{p,q,1,1} \frown Beta(p, q)$, with $f_{p,q,1,1}(x) = \dfrac{x^{p-1}(1 - x)^{q-1}}{B(p, q)}\, \mathrm{I}_{(0,1)}(x)$, where as usual $B(p, q) = \displaystyle\int_0^1 x^{p-1}(1 - x)^{q-1}\mathrm{d}x = \dfrac{\Gamma(p)\,\Gamma(q)}{\Gamma(p + q)}$ is Euler's beta function.

7. $X_{p,1,P,1}$, with pdf $f_{p,1,P,1}(x) = C_{p,1,P,1}\, x^{p-1}\,[-\ln(1-x)]^{P-1}\,\mathrm{I}_{(0,1)}(x)$, where $C_{p,1,P,1} = \dfrac{1}{\int_0^1 x^{p-1}\,[-\ln(1-x)]^{P-1}\mathrm{d}x}$. For $p \in \mathbb{N}$, $C_{p,1,P,1} = \dfrac{1}{\sum_{k=1}^{p}(-1)^{k+1}\binom{p-1}{k-1}\dfrac{\Gamma(P)}{k^P}}$.
8. $X_{1,q,P,1}$, with pdf $f_{1,q,P,1}(x) = \dfrac{q^P}{\Gamma(P)}(1-x)^{q-1}\,[-\ln(1-x)]^{P-1}\,\mathrm{I}_{(0,1)}(x)$.
9. $X_{p,1,1,Q}$, with pdf $f_{p,1,1,Q}(x) = \dfrac{p^Q}{\Gamma(Q)}x^{p-1}\,[-\ln x]^{Q-1}\,\mathrm{I}_{(0,1)}(x)$, that for $Q \in \mathbb{N}$ is the product of Q iid $Beta(p,1)$, i.e. standard uniform random variables raised to the power $\frac{1}{p}$, cf. also Arnold *et al.* [2].
10. ...

(we postpone the discussion of the more complicated models 10-15 to the full paper, since they are not discussed in this shorter version; observe also that the only models for which an explicit evaluation of raw and of central moments is straightforward are those with $q = P = 1$ or with $P = Q = 1$, and so they are the natural candidates to model population dynamics).

3 Population Dynamics, $BeTaBoOp(p,q,P,Q)$ and extreme value models

Brilhante *et al.* [3] used differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,N(t)\,[-\ln[N(t)]]^{1+\gamma} \tag{4}$$

obtaining as solutions the three extreme value models for maxima, Weibull when $\gamma < 0$, Gumbel when $\gamma = 0$ and Fréchet when $\gamma > 0$. The result for $\gamma = 0$ has also been presented in Tsoularis [14] and in Waliszewski and Konarski [19], where as usual in population growth context the Gumbel distribution is called Gompertz function. Brilhante *et al.* [3] have also shown that the associated difference equations

$$x_{n+1} = \alpha\, x_n\,[-\ln x_n]^{1+\gamma},$$

exhibit bifurcation and ultimate chaos, when numerical root finding using the fixed point method, when $\alpha = \alpha(\gamma)$ increases beyond values maintaining the absolute value of the derivative limited by 1.

On the other hand, if instead of the right hand side $N(t)\,[-\ln[N(t)]]^{1+\gamma}$ associated to the $BeTaBoOp(2,1,1,2+\gamma)$ we use as right hand side $[-\ln[1-N(t)]]^{1+\gamma}\,[1-N(t)]$, associated to the $BeTaBoOp(1,2+\gamma,2,1)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,[-\ln[1-N(t)]]^{1+\gamma}\,[1-N(t)]$$

the solutions obtained are the corresponding extreme value models for minima (and bifurcation and chaos when solving the associated difference equations using the fixed point method). In view of the duality of extreme order statistics for maxima and for minima, in the sequel we shall restrict our observation to the case (4) and the associated $BeTaBoOp(2,1,1,2+\gamma)$ model.

As $-\ln N(t) = \sum_{k=1}^{\infty} \frac{[1-N(t)]^k}{k} > 1-N(t)$, for the same value of the malthusian instantaneous growth parameter r we have $r\,N(t)\,[1-N(t)] < r\,N(t)\,(-\ln[N(t))]$, and hence while (1) models sustainable growth in view of the available resources, (4) models extreme value, arguably destructive unsustained growth — for instance cell growth in tumours.

The connection to extreme value theory suggests further observations:

Assume that U_1, U_2, U_3, U_4 are independent identically distributed standard uniform random variables.

1. The pdf of $\min(U_1,U_2)$ is $f_{\min(U_1,U_2)}(x) = 2\,(1-x)\,\mathrm{I}_{(0,1)}(x)$ and the pdf of $\max(U_1,U_2)$ is $f_{\max(U_1,U_2)}(x) = 2\,x\,\mathrm{I}_{(0,1)}(x)$. Hence the $Beta(2,2) \equiv BeTaBoOp(2,2,1,1)$ tied to the Verhults model (1) is proportional to the product of the pdf of the maximum and the pdf of the minimum of independent standard uniforms.
2. The pdf of the product U_3U_4 is $f_{(U_3U_4)}(x) = -\ln\, x\,\mathrm{I}_{(0,1)}(x)$ — and more generally, the pdf of n independent standard uniform random variables is a $BeTaBoOp(1,1,1,n)$ — and hence the pdf of the $BeTaBoOp(2,1,1,n)$ tied to (4) is proportional to the product of $f_{\max(U_1,U_2)}$ by $f_{(U_3U_4)}$. Interpreting $f_{\max(U_1,U_2)}\,f_{(U_3U_4)}$ and $f_{\max(U_1,U_2)}\,f_{\min(U_1,U_2)}$ as "likelihoods", this shows that the model (4) favors more extreme population growth than the model (1).
   More explicitly, the probability density functions $f_{1,1,1,2}f_{(U_3U_4)}(x) = -ln\,x\,\mathrm{I}_{(0,1)}(x)$ and $f_{1,2,1,1}f_{\min(U_1,U_2)}(x) = 2\,(1-x)\,\mathrm{I}_{(0,1)}(x)$ intersect each other at $x \approx 0.203188$, and scrutiny of the graph shows that the probability that $U_3\,U_4$ takes on very small values below that value is much higher than the probability of $\min(U_1,U_2) < 0.203188$, and therefore the controlling retroaction tends to be smaller, allowing for unsustainable growth.
   For more on product of functions of powers of products of independent standard uniform random variables, cf. Brilhante *et al.* [4] and Arnold *et al.* [2].
3. Rachev and Resnick, [11] developed a theory of stable limits of randomly stopped maxima with geometric subordinator (also called geo-max stability) similar to what had been independently achieved by Rényi [12], Kovalenko [7] and in all generality by Kozubowski [8], for a panorama cf. also Gnedenko and Korolev, [6].
   The geo-stable maxima laws are the logistic, the log-logistic and the simetrized log-logistic (corresponding to the Gumbel, Fréchet and Weibull when there is no geometric thinning, and with similr characterization

of domains of attraction). Hence, the classical Verhulst (1) population growth model can also be looked at as an extreme value model, but in a context where there exists a natural thinning that maintains sustainable growth.

More involved population dynamics growth differential equation models do have explicit solution for special combinations of the shape parameters, for instance the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = r\,[N(t)]^{2-\gamma}\left[1-\frac{N(t)}{K}\right]^{\gamma}, \quad \gamma < 2 \tag{5}$$

is

$$N(t) = \frac{K}{1+\left\{(\gamma-1)\,r\,K^{1-\gamma}t+\left(\frac{K}{N_0}-1\right)^{1-\gamma}\right\}^{\frac{1}{1+\gamma}}}$$

as shown by Turner *et al.* [15], cf. also Tsoularis [14].

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