Multivariate maxima of moving multivariate maxima

Helena Ferreira [1] *Department of Mathematics, University of Beira Interior, Covilhã, Portugal*

Abstract: We define a class of multivariate maxima of moving multivariate maxima, generalising the M4 processes. For these stationary multivariate time series we characterise the joint distribution of extremes and compute the multivariate extremal index. We derive the bivariate upper tail dependence coefficients and the extremal coefficient of the new limiting multivariate extreme value distributions.

Keywords: moving multivariate maxima, multivariate extremal index, tail dependence, multivariate extreme value distribution.

1 Introduction

Let $\{\mathbf{Z}_{l,n} = (Z_{l,n,1}, \ldots, Z_{l,n,d})\}_{l\geq 1, -\infty<n<\infty}$ be an array of independent random vectors with standard Fréchet margins and common copula $C_{\mathbf{Z}}$. A multivariate maxima of moving multivariate maxima (henceforth M5) process is defined by

$$Y_{n,j} = \max_{l\geq 1} \max_{-\infty<k<+\infty} \alpha_{lkj} Z_{l,n-k,j},\ j = 1, \ldots, d,\ n \geq 1, \tag{1}$$

where $\{\alpha_{lkj}, l \geq 1, -\infty < k < \infty, 1 \leq j \leq d\}$ are nonnegative constants satisfying

$$\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \alpha_{lkj} = 1, \quad \text{for } j = 1, \ldots, d.$$

When $Z_{l,n,j} = Z_{l,n}$, $j = 1, ..., d$, the M5 process is the M4 process considered by [11].

The common distribution $F_{\mathbf{Y}}$ of $\mathbf{Y}_n = (Y_{n,1}, \ldots, Y_{n,d})$ satisfies

$$F_{\mathbf{Y}}(y_1, ..., y_d) = \prod_{l=1}^{\infty} \prod_{k=-\infty}^{\infty} F_{\mathbf{Z}} \left(\frac{y_1}{\alpha_{lk1}}, ..., \frac{y_d}{\alpha_{lkd}} \right),\ \ y_j > 0,\ j = 1, ..., d,$$

and the relation for the corresponding copulas is

$$C_{\mathbf{Y}}(u_1, ..., u_d) = \prod_{l=1}^{\infty} \prod_{k=-\infty}^{\infty} C_{\mathbf{Z}} \left(u_1^{\alpha_{lk1}}, ..., u_d^{\alpha_{lkd}} \right),\ \ u_j \in [0, 1],\ j = 1, ..., d.$$

This paper is concerned with the extreme value properties of the class of stationary processes defined in (1). The M5 processes contribute to the modelling of variables moving together and generalise the important class of moving maxima discussed by several authors ([3], [2], [5], [11], [12], among others). This is the motivation of this paper which is organised as follows.

By choosing $C_{\mathbf{Z}}$ in the domain of attraction of a max-stable copula C^* we will find a new class of limiting multivariate extreme value (MEV) distribution H for the vector $\mathbf{M}_n = (M_{n1}, \ldots, M_{nd})$ of componentwise maxima from $\mathbf{Y}_1, ..., \mathbf{Y}_n$.

In the third section, we derive the multivariate extremal index of the M5 processes and illustrate the result with some choices C^*.

Finally, we compare the bivariate upper tail dependence coefficients of $F_{\mathbf{Y}}$ with the ones of the limiting MEV distribution H.

[1] E-mail:helenaf@ubi.pt

2 Domains of max-attraction

Let $\{\hat{\mathbf{Y}}_n\}_n \geq 1$ be a sequence of independent random vectors associated to $\{\mathbf{Y}_n\}_n \geq 1$, that is such that $F_{\hat{\mathbf{Y}}_n} = F_{\mathbf{Y}}$, and $\hat{\mathbf{M}}_n = \left(\hat{M}_{n1}, \ldots, \hat{M}_{nd}\right)$ be the corresponding vector of pointwise maxima. Following [11], we present in this section the limiting distributions of the normalised vectors $\hat{\mathbf{M}}_n$ and $\mathbf{M}_n$.

Proposition 2.1 *If $C_{\mathbf{Z}}$ is in the domain of attraction of C^*, that is,*

$$C^n_{\mathbf{Z}}(u_1^{1/n}, ..., u_d^{1/n}) \underset{n\to\infty}{\longrightarrow} C^*(u_1, ..., u_d),$$

then $C_{\mathbf{Y}}$ is in the domain of attraction of

$$\hat{C}(u_1, ..., u_d) = \prod_{l=1}^{\infty} \prod_{k=-\infty}^{\infty} C^* \left(u_1^{\alpha_{lk1}}, ..., u_d^{\alpha_{lkd}}\right) \tag{2}$$

Proof. We have

$$\lim_{n\to\infty} C^n_{\mathbf{Y}}(u_1^{1/n}, ..., u_d^{1/n}) = \lim_{n\to\infty} \prod_{l=1}^{\infty} \prod_{k=-\infty}^{\infty} C^n_{\mathbf{Z}} \left(u_1^{\alpha_{lk1}/n}, ..., u_d^{\alpha_{lkd}/n}\right)$$

and we can take the product of limits since

$$\lim_{n\to\infty} \prod_{l=1}^{\infty} \prod_{k=-\infty}^{\infty} C^n_{\mathbf{Z}} \left(u_1^{\alpha_{lk1}/n}, ..., u_d^{\alpha_{lkd}/n}\right) = exp\left(-\lim_{n\to\infty} \sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} -\ln C^n_{\mathbf{Z}} \left(u_1^{\alpha_{lk1}/n}, ..., u_d^{\alpha_{lkd}/n}\right)\right),$$

where

$$\sum_{k=-\infty}^{\infty} -\ln C^n_{\mathbf{Z}} \left(u_1^{\alpha_{lk1}/n}, ..., u_d^{\alpha_{lkd}/n}\right) \leq \sum_{k=-\infty}^{\infty} \left(-\ln \prod_{j=1}^{d} u_j^{\alpha_{lkj}/n}\right)^n \leq \sum_{k=-\infty}^{\infty} \sum_{j=1}^{d} \alpha_{lkj}(-\ln u_j),$$

which is summable in l, and

$$\ln C^n_{\mathbf{Z}} \left(u_1^{\alpha_{lk1}/n}, ..., u_d^{\alpha_{lkd}/n}\right) \leq \sum_{j=1}^{d} \alpha_{lkj}(-\ln u_j),$$

which is summable in k. It then follows, from the Domminated Convergence Theorem, that

$$\begin{aligned} &\lim_{n\to\infty} \prod_{l=1}^{\infty} \prod_{k=-\infty}^{\infty} C^n_{\mathbf{Z}} \left(u_1^{\alpha_{lk1}/n}, ..., u_d^{\alpha_{lkd}/n}\right) \\ = \quad & exp\left(\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \ln\left(\lim_{n\to\infty} C^n_{\mathbf{Z}} \left(u_1^{\alpha_{lk1}/n}, ..., u_d^{\alpha_{lkd}/n}\right)\right)\right) = \hat{C}\left(u_1, ..., u_d\right). \end{aligned}$$

□

We have then, for $\forall \boldsymbol{\tau} = (\tau_1, \ldots, \tau_d) \in \mathbb{R}^d_+$,

$$P\left(\hat{M}_{n1}\leq\frac{n}{\tau_1},\ldots,\hat{M}_{nd}\leq\frac{n}{\tau_d}\right)=C_{\mathbf{Y}}^n\left(e^{-\frac{\tau_1}{n}},\ldots,e^{\frac{\tau_d}{n}}\right)$$

$$\underset{n\to\infty}{\longrightarrow}\quad\hat{C}\left(e^{-\tau_1},\ldots,e^{-\tau_d}\right)=\prod_{l=1}^{\infty}\prod_{k=-\infty}^{\infty}C^*\left(e^{-\tau_1\alpha_{lk1}},...,e^{-\tau_d\alpha_{lkd}}\right).$$

Let $\hat{H}$ denote the multivariate extreme value distribution with standard Fréchet margins and copula $\hat{C}$, i.e. $\hat{H}(x_1,...,x_d)=\prod_{l=1}^{\infty}\prod_{k=-\infty}^{\infty}C^*\left(e^{-x_1^{-1}\alpha_{lk1}},...,e^{-x_d^{-1}\alpha_{lkd}}\right)$. We will now consider the corresponding limit H for the vector $\mathbf{M}_n=(M_{n1},\ldots,M_{nd})$ of componentwise maxima from $\mathbf{Y}_1,...,\mathbf{Y}_n$ and we will find $H(x_1,...,x_d)=\prod_{l=1}^{\infty}C^*\left(e^{-\max_{-\infty\leq k\leq+\infty}\alpha_{lk1}x_1^{-1}},...,e^{-\max_{-\infty\leq k\leq+\infty}\alpha_{lkd}x_d^{-1}}\right)$.

Proposition 2.2 *If $C_{\mathbf{Z}}$ is in the domain of attraction of C^* then*

$$\lim_{n\to\infty}P\left(M_{n1}\leq\frac{n}{\tau_1},\ldots,M_{nd}\leq\frac{n}{\tau_d}\right)=\prod_{l=1}^{\infty}C^*\left(e^{-\max_{-\infty\leq k\leq+\infty}\alpha_{lk1}\tau_1},...,e^{-\max_{-\infty\leq k\leq+\infty}\alpha_{lkd}\tau_d}\right).\tag{3}$$

Proof. It holds

$$\begin{aligned}
&\lim_{n\to\infty}P\left(M_{n1}\leq\frac{n}{\tau_1},\ldots,M_{nd}\leq\frac{n}{\tau_d}\right)\\
&=\lim_{n\to\infty}\prod_{l=1}^{\infty}\prod_{k=-\infty}^{\infty}C_{\mathbf{Z}}\left(e^{-\max_{1-m\leq k\leq n-m}\alpha_{lk1}\tau_1/n},...,e^{-\max_{1-m\leq k\leq n-m}\alpha_{lkd}\tau_d/n}\right),\\
&=\exp\left(\sum_{l=1}^{\infty}\lim_{n\to\infty}\sum_{k=-\infty}^{\infty}\frac{1}{n}\ln C_{\mathbf{Z}}^n\left(e^{-\max_{1-m\leq k\leq n-m}\alpha_{lk1}\tau_1/n},...,e^{-\max_{1-m\leq k\leq n-m}\alpha_{lkd}\tau_d/n}\right)\right),
\end{aligned}$$

since we can change the first two limits by using analogous arguments to those used in the first result.

Now we have to prove that

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{k=-\infty}^{\infty}\frac{1}{n}\ln C_{\mathbf{Z}}^n\left(e^{-\max_{1-m\leq k\leq n-m}\alpha_{lk1}\tau_1/n},...,e^{-\max_{1-m\leq k\leq n-m}\alpha_{lkd}\tau_d/n}\right)\\
&=\ln C^*\left(e^{-\max_{-\infty\leq k\leq+\infty}\alpha_{lk1}\tau_1},...,e^{-\max_{-\infty\leq k\leq+\infty}\alpha_{lkd}\tau_d}\right)
\end{aligned}$$

For fixed l, let $b_{lkj}=a_{lkj}\tau_j$, $j=1,...,d$, and $b_{lk}=\max_{j=1,\ldots,d}b_{lkj}$ (which are summable in k and l). Suppose that b_{lkj} is maximazed when $k=k^*(l,j)$, b_{lk} is maximazed when $k=k^*(l)$ (not necessarily unique) and assume, without loss of generality, that $k^*(l,1)\leq...\leq k^*(l,d)$. Break the above sum into three sums $S_n^{(i)}$, $1=1,2,3$, corresponding to $1-\min\{k^*(l,1),k^*(l)\}\leq m\leq n-\max\{k^*(l,d),k^*(l)\}$, $m<1-\min\{k^*(l,1),k^*(l)\}$ and $m>n-\max\{k^*(l,d),k^*(l)\}$.

For large n we have

$$S_n^{(1)} = \frac{1}{n} \sum_{m=1-\min\{k^*(l,1),k^*(l)\}}^{n-\max\{k^*(l,d),k^*(l)\}} \ln C_{\mathbf{Z}}^n \left(e^{-\max_{1-m\leq k\leq n-m} \alpha_{lk1}\tau_1/n}, ..., e^{-\max_{1-m\leq k\leq n-m} \alpha_{lkd}\tau_d/n} \right)$$

$$= \frac{n-\max\{k^*(l,d),k^*(l)\}+\min\{k^*(l,1),k^*(l)\}}{n} \ln C_{\mathbf{Z}}^n \left(e^{-\alpha_{lk^*(l,1)1}\tau_1/n}, ..., e^{-\alpha_{lk^*(l,d)d}\tau_d/n} \right),$$

which converges to

$$\ln C^* \left(e^{-\alpha_{lk^*(l,1)1}\tau_1}, ..., e^{-\alpha_{lk^*(l,d)d}\tau_d} \right) = \ln C^* \left(e^{-\max_{-\infty\leq k\leq +\infty} \alpha_{lk1}\tau_1}, ..., e^{-\max_{-\infty\leq k\leq +\infty} \alpha_{lkd}\tau_d} \right).$$

Otherwise,

$$S_n^{(2)} \leq \frac{1}{n} \sum_{m<1-k^*(l)} \max_{1-m\leq k\leq n-m} \max_{j=1,\ldots,d} \alpha_{lkj}\tau_j$$

$$+ \quad \frac{1}{n} \sum_{1-k^*(l)\leq m<1-\min\{k^*(l,1),k^*(l)\}} \max_{1-m\leq k\leq n-m} \max_{j=1,\ldots,d} \alpha_{lkj}\tau_j$$

$$\leq \quad \frac{1}{n} \sum_{m<1-k^*(l)} \max_{1-m\leq k\leq n-m} \max_{j=1,\ldots,d} \alpha_{lkj}\tau_j + \frac{1}{n} \left(\min\{k^*(l,1),k^*(l)\} - k^*(l)\right) b_{l,k^*(l)}.$$

The second term tends to zero and the same holds for the first, by Lemma 3.2 in [11]. The treatment for the sum $S_n^{(3)}$ is analogous (to break the sum accordingly to $n-\max\{k^*(l,d),k^*(l)\} < m < n-k^*(l)$ and $m \geq n-k^*(l)$).

□

For the particular case of $C^*(u_1,...,u_d) = \min_{1\leq j\leq d} u_j$, $(u_1,\ldots,u_d) \in [0,1]^d$, the above result agrees with (3.5) in [11], that is

$$\lim_{n\to\infty} P\left(M_{n1} \leq \frac{n}{\tau_1}, \ldots, M_{nd} \leq \frac{n}{\tau_d} \right) = \exp\left\{ -\sum_{l=1}^{\infty} \max_{-\infty<k<\infty} \max_{1\leq j\leq d} \alpha_{lkj}\tau_j \right\}.$$

If we choose $C^*(u_1,...,u_d) = \prod_{j=1}^d u_j$, $(u_1,\ldots,u_d) \in [0,1]^d$, we find again the result in case two of the last example in [8],

$$\lim_{n\to\infty} P\left(M_{n1} \leq \frac{n}{\tau_1}, \ldots, M_{nd} \leq \frac{n}{\tau_d} \right) = \exp\left\{ -\sum_{j=1}^{d} \sum_{l=1}^{\infty} \max_{-\infty<k<\infty} \alpha_{lkj}\tau_j \right\}.$$

In this last work it is considered only $C_{\mathbf{Z}} = C^*$ and these two cases are treated directly.

By calculating the limits in (2) and (3) for a given initial copula C^*, the above propositions enable us to obtain a larger class of MEV models.

Some choices for C^* are available in the literature. For instance, [1] gives sufficient conditions for an Archimedean copula $C_{\mathbf{Z}}$ to be in the domain of attraction of the Gumbel-Hougaard or logistic copula $C^*(u_1,...,u_d) = \exp\left(-\left(\sum_{j=1}^d (-\ln u_j)^\alpha \right)^{1/\alpha} \right)$. More examples of $C_{\mathbf{Z}}$ in the domain of attraction of a MEV copula C^* can be found in [7] and [4].

3 The multivariate extremal index of M5 process

In this section we will extend the results about the multivariate extremal index of the M4 process, as a corollary of the propositions 2.1 and 2.2.

We first recall the definition of this function $\theta(\boldsymbol{\tau}) = \theta(\tau_1, \dots, \tau_d)$, $\boldsymbol{\tau} \in \mathbb{R}_+^d$, that relates the MEV distribution functions H and $\hat{H}$ and which was introduced by [9].
A $d-$dimensional stationary sequence $\{\mathbf{Y}_n\}$ is said to have a multivariate extremal index $\theta(\boldsymbol{\tau}) \in [0,1]$ $\boldsymbol{\tau} \in \mathbb{R}_+^d$, if for each $\boldsymbol{\tau} = (\tau_1, \dots, \tau_d)$ in $\mathbb{R}_+^d$, there exists $\mathbf{u}_n^{(\boldsymbol{\tau})} = (u_{n1}^{(\tau_1)}, \dots, u_{nd}^{(\tau_d)})$, $n \geq 1$, satisfying

$$nP(Y_{1j} > u_{nj}^{(\tau_j)}) \underset{n\to\infty}{\longrightarrow} \tau_j,\ j = 1, \dots, d,$$

$$P(\hat{\mathbf{M}}_n \leq \mathbf{u}_n^{(\boldsymbol{\tau})}) \underset{n\to\infty}{\longrightarrow} \hat{\gamma}(\boldsymbol{\tau}) \ \textit{and} \ P(\mathbf{M}_n \leq \mathbf{u}_n^{(\boldsymbol{\tau})}) \underset{n\to\infty}{\longrightarrow} \hat{\gamma}(\boldsymbol{\tau})^{\theta(\boldsymbol{\tau})}.$$

Proposition 3.1 *If $C_{\mathbf{Z}}$ is in the domain of attraction of C^* then the multivariate extremal index of the M5 process $\{\mathbf{Y}_n\}$ defined in (1) is given by*

$$\theta(\tau_1, \dots, \tau_d) = \frac{\sum_{l=1}^{\infty} \ln C^* \left(e^{-\max_{-\infty \leq k \leq +\infty} \alpha_{lk1}\tau_1}, \dots, e^{-\max_{-\infty \leq k \leq +\infty} \alpha_{lkd}\tau_d} \right)}{\sum_{l=1}^{\infty} \sum_{k=-\infty}^{\infty} \ln C^* \left(e^{-\alpha_{lk1}\tau_1}, \dots, e^{-\alpha_{lkd}\tau_d} \right)}$$

and the extremal index of Y_{nj} is

$$\theta_j = \sum_{l=1}^{\infty} \max_{-\infty < k < \infty} \alpha_{lkj},\ j = 1, \dots, d.$$

Therefore the copulas C^*, $\hat{C}$ and the copula C of the limiting MEV distribution H are related by a multivariate extremal index $\theta(\tau_1, \dots, \tau_d)$ through

$$C(u_1, \dots, u_d) = \left(\hat{C}(u_1^{1/\theta_1}, \dots, u_d^{1/\theta_d}) \right)^{\theta(-\frac{\ln u_1}{\theta_1}, \dots, -\frac{\ln u_d}{\theta_d})} = \prod_{l=1}^{\infty} \prod_{k=-\infty}^{\infty} \left(C^*(u_1^{\alpha_{lk1}/\theta_1}, \dots, u_d^{\alpha_{lkd}/\theta_d}) \right)^{\theta(-\frac{\ln u_1}{\theta_1}, \dots, -\frac{\ln u_d}{\theta_d})}.$$

The proposition 3.1 leads to the results of [11] and [8] when C^* is the copula of the minimum and product, respectively.
Otherwise, if we take for instance the Logistic copula C^* we find

$$\theta(\tau_1, \dots, \tau_d) = \frac{\sum_{l=1}^{\infty} \left(\sum_{j=1}^{d} \left(\max_{-\infty \leq k \leq +\infty} \alpha_{lkj}\tau_j \right)^{\alpha} \right)^{1/\alpha}}{\sum_{l=1}^{\infty} \sum_{k=\infty}^{\infty} \left(\sum_{j=1}^{d} (\alpha_{lkj}\tau_j)^{\alpha} \right)^{1/\alpha}}.$$

From the spectral measure representation ([10]) of the copula C^*

$$C^*(u_1,...,u_d) = \exp\Big(- \int_{\mathcal{S}_d} \max_{1\le j\le d} w_j(-\ln u_j)\, dW^*(w_1,...,w_d)\Big),$$

where W^* is a finite measure on the unit simplex $\mathcal{S}_d$ in $\mathbb{R}^d_+$ satisfying $\int_{\mathcal{S}_d} w_j\, dW^*(w_1,...,w_d) = 1,\ j = 1,...,d$, we obtain

$$\theta(\tau_1,\ldots,\tau_d) = \frac{\sum_{l=1}^{\infty} \int_{\mathcal{S}_d} \max_{1\le j\le d} \Big(w_j \max_{-\infty\le k\le +\infty} \alpha_{lkj}\tau_j \Big)\, dW^*(w_1,...,w_d)}{\sum_{l=1}^{\infty}\sum_{k=\infty}^{\infty} \int_{\mathcal{S}_d} \max_{1\le j\le d} w_j\alpha_{lkj}\tau_j\, dW^*(w_1,...,w_d)}.$$

4 The tail dependence of M5 process

For a random vector $\mathbf{X} = (X_1,\ldots,X_d)$ with continuos margins $F_1,...,F_d$ and copula C, let the bivariate (upper) tail dependence coefficients parameters be defined by

$$\lambda^{(\mathbf{X})}_{jj'} \equiv \lambda^{(C)}_{jj'} = \lim_{u\uparrow 1} P(F_j(X_j) > u | F_{j'}(X_{j'}) > u),\ 1\le j < j' \le d.$$

It holds

$$\lambda^{(C)}_{jj'} = 2 - \lim_{u\uparrow 1} \frac{\ln C_{jj'}(u,u)}{\ln u},$$

where $C_{jj'}$ is the copula of the sub-vector $(X_j, X_{j'})$.

In the this section we will relate $\lambda^{(C)}_{jj'}$ with $\lambda^{(\hat C)}_{jj'}$. We first remark that, for each $(u_j, u_{j'}) \in [0,1]^2$, we have

$$C_{jj'}(u_j,u_{j'}) = \Big(\hat C_{jj'}(u_j^{1/\theta_j}, u_{j'}^{1/\theta_{j'}})\Big)^{\theta(-\frac{\ln u_j}{\theta_j}, -\frac{\ln u_{j'}}{\theta_{j'}})}, \tag{4}$$

where $\theta(\tau_j,\tau_{j'})$ is the bivariate extremal index of $\{(Y_{nj}, Y_{nj'})\}_{n\ge 1}$. This relation enables us to compare the tail dependence parameters $\lambda^{(C)}_{jj'}$ with $\lambda^{(\hat C)}_{jj'}$ through the function $\theta(\tau_j,\tau_{j'})$.

Proposition 4.1 *If C and $\hat C$ satisfy (4) then*

(a) $\lambda^{(C)}_{jj'} = 2 + \theta(\frac{1}{\theta_j}, \frac{1}{\theta_{j'}}) \ln \hat C_{jj'}(e^{-1/\theta_j}, e^{-1/\theta_{j'}})$,

(a) $\lambda^{(C)}_{jj'} = \lambda^{(\hat C)}_{jj'} + \ln \frac{\hat C_{jj'}(e^{-\theta(\frac{1}{\theta_j},\frac{1}{\theta_{j'}})/\theta_j}, e^{-\theta(\frac{1}{\theta_j},\frac{1}{\theta_{j'}})/\theta_{j'}})}{\hat C_{jj'}(e^{-1}, e^{-1})}$.

Proof. From the spectral measure representation of the copula $\hat C$, with measure $\hat W$, we get

$$\lambda^{(\hat C)}_{jj'} = 2 - \lim_{u\uparrow 1} \frac{\ln \hat C_{jj'}(u,u)}{\ln u} = 2 - \int_{\mathcal{S}_d} \max\{w_j, w_{j'}\}\, d\hat W(w_1,...,w_d) = 2 + \ln \hat C_{jj'}(e^{-1}, e^{-1}). \tag{5}$$

By using (4) and the homogeneity of order 0 of the multivariate extremal index, it follows that

$$\lambda_{jj'}^{(C)} = 2 - \lim_{u\uparrow 1}\frac{\ln C_{jj'}(u,u)}{\ln u} = 2 - \lim_{u\uparrow 1}\theta(\frac{-\ln u}{\theta_j},\frac{-\ln u}{\theta_{j'}})\frac{\ln \hat{C}_{jj'}(u_j^{1/\theta_j},u_{j'}^{1/\theta_{j'}})}{\ln u}$$

$$= 2-\theta(\frac{1}{\theta_j},\frac{1}{\theta_{j'}})\lim_{u\uparrow 1}\frac{\int_{\mathcal{S}_d}\max\{\frac{-\ln uw_j}{\theta_j},\frac{-\ln uw_{j'}}{\theta_{j'}}\}\,d\hat{W}(w_1,...,w_d)}{-\ln u}$$

$$= 2-\theta(\frac{1}{\theta_j},\frac{1}{\theta_{j'}})\int_{\mathcal{S}_d}\max\{\frac{w_j}{\theta_j},\frac{w_{j'}}{\theta_{j'}}\}\,d\hat{W}(w_1,...,w_d)$$

$$= 2+\theta(\frac{1}{\theta_j},\frac{1}{\theta_{j'}})\ln\hat{C}_{jj'}(e^{-1/\theta_j},e^{-1/\theta_{j'}}),$$

which has (5) as a particular case. To obtain the second statement we combine the first with (5) and use the max-stability of $\hat{C}_{jj'}$, as follows:

$$\lambda_{jj'}^{(C)} = 2+\ln\hat{C}_{jj'}(e^{-1},e^{-1})-\ln\hat{C}_{jj'}(e^{-1},e^{-1})+\theta(\frac{1}{\theta_j},\frac{1}{\theta_{j'}})\ln\hat{C}_{jj'}(e^{-1/\theta_j},e^{-1/\theta_{j'}})$$

$$= \lambda_{jj'}^{(\hat{C})}+\ln\frac{\left(\hat{C}_{jj'}(e^{-1/\theta_j},e^{-1/\theta_{j'}})\right)^{\theta(\frac{1}{\theta_j},\frac{1}{\theta_{j'}})}}{\hat{C}_{jj'}(e^{-1},e^{-1})}$$

$$= \lambda_{jj'}^{(\hat{C})}+\ln\frac{\hat{C}_{jj'}(e^{-\theta(\frac{1}{\theta_j},\frac{1}{\theta_{j'}})/\theta_j},e^{-\theta(\frac{1}{\theta_j},\frac{1}{\theta_{j'}})/\theta_{j'}})}{\hat{C}_{jj'}(e^{-1},e^{-1})}.$$

□

We will now apply these relations to the particular case of M5 processes, enhancing the effect of the multivariate extremal index in the tail dependence.

Proposition 4.2 *Let $\{\mathbf{Y}_n\}_n \geq 1$ be a M5 process defined as in (1) and such that $C_{\mathbf{Z}}$ is in the domain of attraction of C^*. Then, for any $1 \leq j < j' \leq d$ it holds*

(a) $\lambda_{jj'}^{(\hat{C})} = 2+\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\ln C^*_{jj'}\left(e^{-\alpha_{lkj}},e^{-\alpha_{lkj'}}\right).$

(b) $\lambda_{jj'}^{(C)} = 2+\theta(\frac{1}{\theta_j},\frac{1}{\theta_{j'}})\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\ln C^*_{jj'}\left(e^{-\alpha_{lkj}/\theta_j},e^{-\alpha_{lkj'}/\theta_{j'}}\right).$

Proof. By using (5) and the Proposition 2.1, we obtain (a); by using the Proposition (4.1)-(a) and Proposition 2.1, we get (b). □

If $C^*(u_1,...,u_d)=\min_{1\leq j\leq d}u_j$, $(u_1,\ldots,u_d)\in[0,1]^d$, we obtain

$$\lambda_{jj'}^{(\hat{C})} = 2-\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\max\{\alpha_{lkj},\alpha_{lkj'}\} \tag{6}$$

in general greater than zero, which agrees with the result (2.10) presented in [6]. We here also say from (b) that, for this copula,

$$\lambda_{jj'}^{(C)} = 2-\sum_{l=1}^{\infty}\max\left\{\max_{-\infty\leq k\leq+\infty}\alpha_{lkj}/\theta_j,\max_{-\infty\leq k\leq+\infty}\alpha_{lkj'}/\theta_{j'}\right\}$$

and it holds $\lambda_{jj'}^{(C)} > \lambda_{jj'}^{(\hat{C})}$ if and only if

$$\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\max\{\alpha_{lkj},\alpha_{lkj'}\} - \sum_{l=1}^{\infty}\max\left\{\max_{-\infty\leq k\leq+\infty}\alpha_{lkj}/\theta_j,\ \max_{-\infty\leq k\leq+\infty}\alpha_{lkj'}/\theta_{j'}\right\} > 0,$$

where $\theta_j = \sum_{l=1}^{\infty}\max_{-\infty<k<\infty}\alpha_{lkj},\ j=1,\ldots,d.$

We can easily construct examples of M4 processes for which $\lambda_{jj'}^{(C)} > \lambda_{jj'}^{(\hat{C})}$ or $\lambda_{jj'}^{(C)} < \lambda_{jj'}^{(\hat{C})}$.

Since $\lambda_{jj'}^{(\hat{C})} = \lambda_{jj'}^{(C_{\mathbf{Y}})}$, the result in (6) says that in the M4 processes the variables $Y_{n,j},\ j=1,...,d$, are in general asymptotically dependent. For the M5 processes we can choose C^* in order to produce variables asymptotically independent. Take for instance C^* such that $\lambda_{jj'}^{(C^*)} = 0$ and $\alpha_{lkj} = \alpha_{lkj'}$. We then find $\lambda_{jj'}^{(\hat{C})} = \lambda_{jj'}^{(C^*)} = 0$.

The Proposition 4.2. also points out that even for a choice of a copula C^* with symmetric tail dependencies, the values of the signatures α_{lkj} can lead to copulas C and $\hat{C}$ with asymmetric tail dependencies.

The results in the above proposition are translations of the classical result $\lambda = 2-\epsilon$ for $C_{jj'}$ and $\hat{C}_{jj'}$. In fact, if we define the extremal coefficient of the MEV copula C as the constant ϵ_C such that $C(u,...,u) = u^{\epsilon_C}$ for all $u\in[0,1]$, then from

$$C(u_1,...,u_d) = \exp\left(-\theta(\frac{\ln u_1}{\theta_1},...,\frac{\ln u_d}{\theta_d})\int_{\mathcal{S}_d}\max_{1\leq j\leq d}\frac{-\ln u_j w_j}{\theta_j}\,d\hat{W}(w_1,...,w_d)\right),$$

we find

$$\epsilon_C = -\theta(\frac{1}{\theta_1},...,\frac{1}{\theta_d})\ln\hat{C}\left(e^{-1/\theta_1},...,e^{-1/\theta_d}\right)$$

and, in particular,

$$\epsilon_{\hat{C}} = -\ln\hat{C}\left(e^{-1},...,e^{-1}\right).$$

The relation between $\hat{C}_{jj'}$ and $C^*_{jj'}$ enables now to obtain the Proposition 4.2. from $\lambda_{jj'}^{(\hat{C})} = 2-\epsilon_{\hat{C}_{jj'}}$ and $\lambda_{jj'}^{(C)} = 2-\epsilon_{C_{jj'}}$.

The results for this class of moving multivariate maxima show us that if we obtain or estimate a tail dependence parameter of the common distribution of the variables in a stationary sequence we don't have necessarily the corresponding parameter in the limiting MEV model C. The multivariate extremal index of the stationary sequence can increase or decrease the tail dependence and the extremal coefficients of the limiting MEV model $\hat{C}$ arising from the i.i.d associated sequence.

Other features of the M5 processes such as a directory of tail dependence coefficients for different C^* and signatures α_{lkj}, illustrating the range of dependence structures, the model selection and estimation are key directions in a future research.

References

[1] Capéraà, P., Fougères, A.L. and Genest, C. (2000) Bivariate distributions with given extreme value attractor. J. Multivariate Anal., 72, 30-49.

[2] Davis, R.A. and Resnick, S.I. (1989) Basic properties and prediction of Max-ARMA processes. Adv. Appl. Prob., 21, 781-803.

[3] Deheuvels, P. (1983) Point processes and multivariate extreme values. J. Multivariate Anal., 13, 257-252.

[4] Demarta, S. and McNeil, A. (2005) The t-copula and related copulas. International Statistical Review 73, 111-129.

[5] Hall, P., Peng, L. and Yao, Q. (2002) Moving-maximum models for extrema of time series. J. Stat. Plan. Infer., 103, 51-63.

[6] Heffernan, J. E., Tawn, J. A., Zhang, Z. (2007). Asymptotically (in)dependent multivariate maxima of moving maxima processes, Extremes, 10, 57-82.

[7] Hüsler, J. and Reiss, R. (1989) Maxima of normal random vectors: between independence and complete dependence. Statist. Probab. Lett., 76, 283-286.

[8] Martins, A.P. and Ferreira, H. (2005) The extremal index and the dependence structure of a multivariate extreme value distribution. Test, Vol. 14, 2, 433-448.

[9] Nandagopalan, S. (1990). Multivariate extremes and estimation of the extremal index. Ph.D. Dissertation, Department of Statistics, University of North Carolina, USA.

[10] Resnick, S. (1987). *Extreme Values, Regular Variation and Point Processes.* Springer-Verlag, New York.

[11] Smith, R.L., Weissman, I. (1996). Characterization and estimation of the multivariate extremal index. Technical Report, Univ. North, Carolina.

[12] Zhang, Z. and Smith, R.L. (2004) The behavior of multivariate maxima of moving maxima processes. J. Appl. Probab., 41, 1113-1123.