Discrete and Continuous Time Extremes of Stationary Processes

K.F. Turkman
CEAUL-FCUL, University of Lisbon

Abstract: In many applications, the primary interest is the supremum of some continuous time process over a specified period. However, data are usually available over a discrete set of times and the inference can only be made for the maximum of the process over this discrete set of times. If we want to estimate the extremes of the continuous time process based on discrete time data, we need to understand the relationship between the continuous and discrete extremes. Thus, we look at asymptotic joint distributions of the maxima of stationary processes and their discrete versions.

AMS subject Classification: 60G70
Key words: Extremes, Stationary processes, Pickands' grid.

This work is partially supported by FCT, POCTI and project PTDC/MAT/64353/2006

1 Introduction

In many applications, the primary interest is the supremum of some continuous time process over a specified period. However, data are usually available over a discrete set of times and the inference can only be made for the maximum of the process over this discrete set of times. The continuous time maximum will be larger than the maxima of discrete versions sampled at different intervals and if we want to estimate the extremes of the continuous time process based on discrete time data, we need to make an adjustment to allow for the effect of discrete sampling and provide a measure of how much smaller it tends to be. In this paper, we make a review of the most relevant results.

Robinson and Tawn(2000) were the first to point out the importance of sampling interval on the extremal properties of a process observed at regular discrete time points. They showed this effect on the extremal indices of the discrete sub-sequences based on the following arguments:

Let $X(t), t \geq 0$ be a stationary process with distribution $F(x)$. For any time interval $[0, T]$, let $X_\delta(i) = X(i\delta), i = 0, 1, .., T/\delta$, be a subsequence observed at spacing δ. Let

$$M(T) = \sup_{t \in [0,T]} X(t),$$

$$M_\delta(T) = \max_{0 \leq i \leq T/\delta} X(i\delta),$$

be the respective maxima of the continuous process and its subsequence sampled at the δ spacing.

If for any two subsequences sampled at spacings δ and ϵ, the maxima converge with suitable normalization, then for large T,

$$P(M_\delta(T) \leq x) \sim F^{[T/\delta]\theta_\delta}(x) = G_\delta(x),$$

$$P(M_\epsilon(T) \leq x) \sim F^{[T/\epsilon]\theta_\epsilon}(x) = G_\epsilon(x),$$

where $\theta_\delta \in (0, 1]$ and $\theta_\epsilon \in (0, 1]$ are extremal indices of the sequences $X_\delta(i\delta)$ and $X_\epsilon(j\epsilon)$. Since

$$G_\epsilon(x) \sim G_\delta^{(\theta_\epsilon \delta)/(\epsilon \theta_\delta)}(x) \tag{1}$$

one can relate the extremal properties of sub-samples at different sampling intervals through the extremal indices. Scotto et al(2003) obtain more precise limit results from a point process formulation which exemplify the findings of Robinson and Tawn(2000) and offer more details for a particular class of time series.

In order to relate the asymptotic distribution of $M(T)$ to $M_\delta(T)$, for some fixed sampling spacing δ, it is tempting to use the above arguments and argue that as $\epsilon \to 0$, if

$$\lim_{\epsilon \to 0} \theta_\epsilon / \epsilon = H,$$

then (taking$\delta = 1$ and $\theta_\delta = \theta$)

$$P(M(T) \leq x) \sim G_1^{H/\theta}(x).$$

In this case, H/θ may be seen as the adjustment needed to allow for using a discrete subsequence.

Based on the assumption that for most continuous processes, there is a fixed sampling interval ϵ, for which $M_\epsilon(T)$ is a sufficiently good approximation to $M(T)$ over the time interval $[0,T]$, Anderson(2003) suggests using

$$\phi = (\theta_\epsilon \delta)/(\epsilon\theta_\delta), \tag{2}$$

as the adjustment for using discrete sampling. He further shows that

$$1 \leq \phi \leq \frac{1}{P(S \geq \delta)}, \tag{3}$$

where S is the excursion time of the continuous process above a high level, which he calls the storm period. The probability that appears as the upper bound in (3) cannot be estimated from the discrete δ-observations, but if one can give a conservative estimate for the probability of storm duration being longer than the sampling interval δ, for example by eliciting expert opinion, then, as Anderson(2003) suggests, the inequality (3) can help to relate important quantities such as the return levels calculated at different sampling intervals. If $x_{n,\epsilon}$, $x_{n,\delta}$ are respectively n time units return levels based on ϵ and δ sampling intervals, then

$$n(1 - G_\delta(x_{n,\delta}) = 1,$$

$$n(1 - G_\epsilon(x_{n,\epsilon}) = n(1 - G_\delta^\phi(x_{n,\epsilon}) = 1,$$

and $x_{n,\epsilon} = x_{n\phi,\delta}$. Thus, if ϵ sampling is sufficiently dense so that $M_\epsilon(h)$ is a sufficiently good approximation to $M(h)$, then $x_{n\phi,\delta}$ can be taken as the n time units return level from the continuous observations.

The above approximations and bounds suggested by Anderson(2003) depend strongly on the assumption that one can approximate a continuous maximum in terms of a discrete maximum over a fixed sampling interval to a desired level of accuracy. Therefore, one needs to understand the relation between the maxima of continuous and discrete time processes in order to judge the robustness of the arguments given by Anderson(2003). Further, such results may help to find sharper bounds for the adjustment.

The coarsest grid over which continuous and discrete maxima over fixed intervals have the same asymptotic distribution, is fundamental in obtaining limit results. Such grid is defined as a family of uniformly-spaced grids with grid spacing converging to zero at a specified rate and is called the Pickands' grid. A standard Pickands' grid relating continuous and discrete maxima over fixed intervals is suggested by Leadbetter et al(1983). See Also Albin(1987,1990) and Piterbarg(2004). Although, This grid adapted to $X(t)$ requires hard to verify technical conditions, it permits elegant characterization of results.

This family of grids(we refer to them by their spacing δ) which will be defined formally in section 2, is taken as a function of the high threshold u such that

$$\delta_a = \delta(a,u) = \{aq(u)j, j = 0, 1, 2...\},$$

where $a > 0$ is an arbitrarily small positive real number and $q(u)$ is a sequence typically converging to 0 as $u \to \infty$. Here, a is a constant regulating the rate of convergence of the grid spacing to 0. As Leadbetter et al (1983) suggest, one can define a universal grid δ without the extra parametrization depending on a, but this extra parametrization brings flexibility in proofs. Note that, as the threshold u tends to infinity, the excursion times of the continuous process above this high level get shorter and in order to capture these events by a discrete version of the process, corresponding grid spacing needs to converge to zero accordingly. $q(u)$ quantifies this relationship.

We will call any other grid $\delta_b = \{jbg(u), j = 0, 1, 2...\}$ a sparse grid if as $u \to \infty$, $q(u) = o(g(u))$, Pickands' grid if if $q(u) = O(g(u))$, and dense grid if $g(u) = o(q(u))$. If δ_a is a Pickands' grid, then the limit as $a \to 0$, δ_a is a dense grid.

Conditions for the existence of the limit over increasing intervals

$$\lim_{T \to \infty} P(M(T) \leq u_T(x)) = G(x),$$

with suitable linear normalization $u_T(x) = a_T + b_T x$ and its relation to maxima over Pickands' grid are well known. In this case, the high level has to be chosen as a function of the increasing interval and consequently, the Pickands' grid is also a function of the increasing time interval. See Leadbetter et al(1983) for Gaussian processes, and Albin(1987, 1990) for general stationary processes. However, the most complete characterization of the relation between discrete and continuous time extremes is given by Piterbarg(2004) for stationary Gaussian processes. See Husler(2004) for the generalization to locally stationary Gaussian processes. Typically, for a grid δ_a with suitable normalizations, the asymptotic joint distribution

$$P(M(T) \leq u_T(x), M_{\delta_a}(T) \leq u_{T,\delta_a}(y)) \tag{4}$$

as $T \to \infty$, is studied for 3 distinct cases:

- When the sampling is done over a dense grid. In this case, (4) is studied for a Pickand's grid δ_a, and as $a \to 0$, the bivariate limiting distribution given in (4) is degenerate, converging to one of the identical marginal distributions of the coinciding continuous and discrete time maxima. Thus

  $$\lim_{a \to 0} \lim_{T \to \infty} P(M(T) \leq u_T(x), M_{\delta_a}(T) \leq u_{T,\delta_a}(y)) = G(z),$$

  where $z = \min(x, y)$.

- If the sampling frequency is at a critical level, that is, when the sampling is done over a Pickands' grid then the maxima are asymptotically dependent, and the limit of (4) for fixed a is given by

  $$G(x) G_{\delta_a}(y) G(a, x, y), \tag{5}$$

  where $G(x)$ and $G_{\delta_a}(y)$ are the respective limiting marginal distributions of the continuous and discrete time maxima and $G(a, x, y)$ is a function explaining the degree of asymptotic dependence of the respective maxima.

- If the sampling frequency is sufficiently low and δ_a in (4) is a sparse grid, then typically the maxima of the continuous and discrete maxima grow with different rates, but it is still possible to find suitable sets of normalization $u_{T,\delta_a}(x)$ and $u_T(y)$, such that the normalized maxima are asymptotically independent, yet having non-degenerate asymptotic marginal distributions.

The function $G(a, x, y)$ in (5) is calculated by Piterbarg(2004) for Gaussian processes. Specifically, if $X(t)$ is a zero mean stationary Gaussian process with unit variance and covariance function $r(t)$ satisfying for some $\alpha > 0$,

$$r(t) = 1 - |t|^\alpha + o(|t|^\alpha),$$

as $t \to 0$ with $r(t) < 1$ for all $t > 0$, and further as $t \to \infty$, $r(t) = o(1/\log t)$ then, with suitable normalization

$$\begin{aligned} & P(M(T) \le u_T(x), M_{\delta_a}(T) \le u_{T,\delta_a}(y)) \\ = \; & G(x) G_{\delta_a}(y) G(a, x, y) \\ = \; & \exp(-e^{-x}) \exp(-e^{-y}) \exp(-G_a(\log H + x, \log H_a + y)). \end{aligned} \tag{6}$$

Here, $0 < G_a(x, y) < \infty$ appears as the limit

$$G_a(x, y) = \lim_{T \to \infty} \frac{1}{T} G_a(x, y, T),$$

with

$$G_a(x, y, T) = \int_{-\infty}^{\infty} e^{\upsilon} P(\max_{k:ka \in [0,T]} \sqrt{2} B_{\alpha/2}(ka) > \upsilon, \max_{k:t \in [0,T]} \sqrt{2} B_{\alpha/2}(t) - t^\alpha > \upsilon + y) d\upsilon. \tag{7}$$

The process $B_{\alpha/2}(t)$ that appears in the expression (7) is the fractional Brownian motion with variance $|t|^\alpha$, whereas H, H_a are Pickands' constants that appear in the marginal limiting distributions. See for example, Leadbetter et al(1983) or Piterbarg(2004).

In the Gaussian case, all asymptotic results on extremes can conveniently be characterized by the covariance function and the proofs are constructed around this tool. For non-Gaussian processes, different sets of conditions are needed. In section 2, we report similar results for the joint asymptotic distribution of continuous-discrete maxima of stationary, but not necessarily Gaussian processes. These results are constructed on the assumptions and techniques of Albin(1987, 1990). We will first look at the asymptotic joint distribution of maxima sampled over a Pickands' grid δ_a and any other grid δ_b, namely

$$P(M_{\delta_a}(h) \le u, M_{\delta_b}(h) \le u'),$$

in a fixed interval $[0, h]$, for some suitably chosen and increasing levels u and u', then extend the results to increasing time intervals. For relative ease in notation, we report the results for stationary processes with regularly varying tails, but following Albin(1990), it is possible to extend the results for other types of tail behavior.

Clearly, the rate at which the Pickands' grid tends to 0 will depend on sample path properties of the continuous process as well as the tail behavior of its marginal distribution. Here we give some examples:

1. If $X(t)$ is a stationary, 0 mean Gaussian process with covariance function

   $$r(t) = 1 - C|t|^{\alpha} + o(|t|^{\alpha}),$$

   for some $\alpha \in (0, 2]$, and $C > 0$, as $t \to 0$, then the Pickands' grid is chosen as

   $$\delta_a = \{jaq(u), j = 0, 1, 2, ..\}$$

   with

   $$q(u) = u^{-2/\alpha},$$

   so that

   $$\lim_{a \downarrow 0} \lim u \to \infty P(M(h) > u) - P(M_{\delta_a}(h) > u)| = 0, \tag{8}$$

   See Piterbarg(2004) Berman, 1992)

2. If $X(t)$ is a standardized differentiable stationary Gaussian process satisfying

   $$r(t) = 1 + \frac{1}{2} r''(0)t^2 + o(t^2),$$

   as $t \to 0$ and $Y(t)$ is the moving $\mathcal{L}^2$-norm process given by

   $$Y(t) = \int_t^{1+t} X^2(s)ds,$$

   then the grid ϵ can be chosen with

   $$q(u) = (1 \vee u)^{-1/2}.$$

   (Albin, 2001).

3. If $X(t)$ is an α-stable process with $\alpha > 1$, then it is possible to take $q(u) = 1$, and (8) will hold with $a \to 0$. (Leadbetter and Hsing,1998 or Samorodnitsky and Taqqu,1994).

4. On the other hand, If $X(t)$ is a moving average of an α-stable process, with $\alpha > 1$, then (8) will hold for a fixed grid ϵ, that is, it will hold with $q(u) = 1$ and for any $a > 0$. (Albin, 2001)

5. If $X(t)$ is an α-stable process with $\alpha < 1$ then (8) holds with

   $$q(u) = (-u)^{\alpha/[2(1-\alpha)]},$$

   as $a \to 0$. (Albin, 2001).

6. If $X_i(t)$ are independent standardized stationary processes with covariance functions satisfying

$$1 + \frac{1}{2}C_i|t|^\alpha + o(|t|^\alpha),$$

as $t \to 0$, for some $C_i > 0$, $\alpha \in (0, 2]$,

$$Z(t) = \sum_{i=1}^{m} X_i^2(t),$$

then (8) will hold with

$$q(u) = u^{-1/\alpha},$$

as $a \to 0$.(Albin, 1987).

It is clear that except for some special processes such as the moving average α-stable process with $\alpha > 1$, it may not be possible to find a fixed sampling interval, for which the discrete maximum approximates the continuous maximum to a desired level. Note that the continuous maximum is almost surely larger than the discrete maximum and the function $q(u)$ also quantifies the relative size of each maxima through the relation

$$q(u) \sim \frac{P(X(0) > u)}{P(M(0,1) > u)},$$

as $u \to \infty$. See Hsing and Leadbetter (1998).

In the next section, we state the technical conditions as well as the main results for the marginal convergence of maxima of stationary process with heavy tailed distributions, which will help in understanding the asymptotic convergence of the joint distributions of continuous anddiscrete time extremes. The proofs will be omitted, as they can be found in Albin(1987, 1990). The conditions and the results will be grouped under subsections, first for results on finite intervals, then on increasing intervals. The proofs of the new results on the asymptotic joint distributions of continuous and discrete time extremes given in Theorems 2 and 5 are tedious, and therefore will not be given here. The detailed arguments can be found in Turkman(2010). In section 3, we give some asymptotic results on the maxima of the periodogram of a Gaussian time series, calculated over fourier frequencies $\omega_j = 2\pi j/n$, $j = 1, 2, ..., [\frac{1}{2}(n-1)]$ and continuous frequencies in $[0, \pi]$ to highlight these technical results.

2 Conditions and Main results

2.1 Finite intervals

Assume that the stationary process $X(t)$ satisfies the following sufficient conditions of Albin(1987,1990) for the marginal convergence of the continuous maximum over finite intervals:

1. *Condition C1*

   F belongs to the Frechét domain of attraction so that for any $x > 0$,

$$\lim_{u\to\infty} \frac{1-F(ux)}{1-F(u)} = x^{-c},$$

   for some $c > 0$.

2. *Condition C2*

   For a strictly positive function $q = q(u)$, let $\delta_a = \{aq(u)j, j = 0, 1, 2, ..., [h/aq]\}$ be a grid over the interval $[0, h]$ such that,(writing for simplicity $q = q(u)$)

$$\limsup_{u\to\infty} \frac{q(u)}{1-F(u)} P(M(h) > u, \max_{a\le aqj\le h} X(aqj) \le u) = 0, \qquad (9)$$

   as $a \to 0$. For any fixed, but sufficiently small $a > 0$, we call δ_a which makes the discrete approximations sufficiently accurate in the sense given in (9) as *Pickands' grid.* On the other hand, any Pickands' grid with $a \to 0$ will be called dense grid.

3. *Condition C3*

   Assume that there exist a sequence of random variables $\{\eta_{a,x}(k)\}_{k=1}^{\infty}$, and a strictly positive function $q(u)$ with $\lim_{u\to\infty} q(u) = 0$ such that for all $x \ge 1$ and for all $a > 0$, and for any finite integer N, as $u \to \infty$,

$$(\frac{1}{u}X(aq), ..., \frac{1}{u}X(aqN)|\frac{1}{u}X(0) > x) \to^{D} (\eta_{a,x}(1), ..., \eta_{a,x}(N)). \qquad (10)$$

4. Condition *C4, Short-lasting-exceedances*

$$\limsup_{u\to\infty} \frac{1}{1-F(u)} \sum_{k=N}^{[h/aq]} P(X(0) > u, X(aqk) > u) \to 0,$$

   as $N \to \infty$, For all fixed $a > 0$.

We refer the reader to Albin(1990) for the details of these assumptions. Condition *C3* is a natural extention of the condition *C1* and it is satisfied by most processes. Albin(1990) gives an alternative condition to *C2* which can be verified by two dimensional distributions of the process. However, we note that condition *C4* is not always satisfied. We refer to Albin(1984) and Husler and Piterbarg(2010) for asymptotic results when this condition is violated.

Theorem 1 : *(Marginal convergence of maxima over Pickands' or denser grids, Albin(1987))*

1. Assume that conditions *C1, C3 and C4* are satisfied. Then for any $a > 0$ fixed,

$$\lim_{u\to\infty} \frac{q(u)}{1-F(u)} P(M_\delta(h) > u) = hH_{a,1}(1),$$

and for any $x > 0$

$$\lim_{u\to\infty} \frac{q(u)}{1-F(ux)} P(M_\delta(h) > ux) = hH_{a,x}(x)$$

where

$$H_{a,x}(x) = \frac{1}{a} P(\max_{k\geq 1} \eta_{a,x} \leq x), \tag{11}$$

and

$$\lim_{a\to 0} H_{a,x}(x) = H_x(x), \tag{12}$$

exist with $0 < H_x(x) < \infty$.

2. If further, condition *C2* is satisfied, then

$$\lim_{u\to\infty} \frac{q(u)}{1-F(u)} P(M(h) > u) = hH_1(1), \tag{13}$$

and

$$\lim_{u\to\infty} \frac{q(u)}{1-F(ux))} P(M(h) > ux) = hH_x(x),$$

so that

$$\lim_{u\to\infty} \frac{q(u)}{1-F(u)} P(M(h) > ux) = hH_x(x)x^{-c}. \tag{14}$$

Note that $H_x(x)$ is not a constant, and therefore (14) may indicate that the distribution functions of $M(h)$ and F may not belong to the same domain of attraction. However, Albin(1990) shows that

$$\frac{q(ux)}{q(u)} = x^{-c^*}, \tag{15}$$

for some $c^* \in [0, c)$, for all $x > 0$ so that as $u \to \infty$,

$$\lim_{u\to\infty} \frac{q(u)}{1-F(u)} P(M(h) > ux) = hH_1(1)x^{-(c-c^*)},$$

for some $c^* \in [0, c)$. Hence, the distribution functions of $M(h)$ and F belong to Frechét domain of attraction, having different shape parameters.

Condition *C3* is given in terms of conditioning on the event $\{X(0) > ux\}$. However, an alternative formulation in terms of conditioning on the event $\{X(0) = ux\}$ can also be given:

Corollary 1

Assume further that F has a density f satisfying

$$\lim_{u\to\infty} \frac{uf(u)}{1-F(u)} = c,$$

for some $c > 0$ and assume further that there exists variables $\{\zeta_{a,x}(k)\}_{k=1}^{\infty}$ such that

$$(\frac{1}{u}X(aq), ..., \frac{1}{u}X(Naq)|X(0) = ux) \to^D \{\zeta_{a,x}(k)\}_{k=1}^{N}, \quad (16)$$

for all $X > 1$ and for all N. Then (13) and (14) hold with

$$H_x(x) = \lim_{a\to 0} \frac{1}{a} \int_1^{\infty} P(\max_{k\geq 1} \zeta_{a,xy}(k) \leq x) c y^{-(c+1)} dy,$$

and

$$H_1(1) = \lim_{a\to 0} \frac{1}{a} \int_1^{\infty} P(\max_{k\geq 1} \zeta_{a,y}(k) \leq 1) c y^{-(c+1)} dy.$$

Equipped with the results for marginal convergence, we can now state the results for joint convergence:

Theorem 2: *(Joint convergence of maxima over Pickands' or denser grids)*

1. For any $a > 0$, $b > 0$, such that $a < b$, Let

   $$\delta_a = \{jaq(u), j = 0, 1, 2, ..., [h/aq]\}$$

   and

   $$\delta_b = \{jbq(u), j = 0, 1, 2, ..., [h/bq]\}$$

   be two Pickands' grids satisfying conditions *C1-C3*. Let

   $z = \min(x, y)(= x \wedge y)$ and $v = \max(x, y)(= x \vee y)$.

   Then

   $$\lim_{u\to\infty} \frac{q(u)}{1 - F(u)} P(\{M_{\delta_a}(h) > ux\} \cup \{M_{\delta_b}(h) > uy\}) = hH_{a,b,z}(x, y)z^{-c},$$

   where

   $$H_{a,b,z}(x, y) = \frac{1}{a} P(\max_{k\geq 1} \eta_{a,z}(k) \leq x, \max_{k\geq 1} \eta_{b,z}(k) \leq y).$$

2. $$\lim_{a\to 0} H_z(a, b, x, y) = H_z(b, x, y),$$

   exists with $0 < H_z(b, x, y) < \infty$ and

   $$\lim_{u\to\infty} \frac{q(u)}{1 - F(u)} P(M(h) > ux \cup M_{\delta_b}(h) > uy) = hH_z(b, x, y)z^{-c},$$

   where

   $$H_z(b, x, y) = \lim_{a\to 0} \frac{1}{a} P(\max_{i\geq 1} \eta_{a,z}(i) \leq x, \max_{j\geq 1} \eta_{b,z}(j) \leq y).$$

3. The limit

$$\lim_{b\to 0} H_z(b,x,y) = H_z(z)$$

exists with $0 < H_z(z) < \infty$, where

$$H_z(z) = \lim_{b\to 0} \frac{1}{b} P(\max_{i\geq 1} \eta_{z,b}(i) \leq z),$$

and hence

$$\lim_{b\to 0} \lim_{u\to\infty} \frac{q(u)}{1-F(u)} P(M(h) > ux \cup M_{\delta_b}(h) > uy) = hH_z(z)z^{-c}.$$

The proof is quite tedious and is based on finding asymptotic bounds for the expression

$$P(\{M_{\delta_a}(I_0) > uy\} \cup \{M_{\delta_b}(I_0) > ux\})$$

as contrast to the proof of Theorem 1 of Albin(1990), where asymptotic bounds for the expression

$$P(\{M_{\delta_a}(I_0) > uy\}$$

are derived. See Turkman(2011) For details.

We now look at the asymptotic independence of maxima calculated over Pickand's and sparse grids.

For some strictly positive function $g = g(u)$ such that $\lim_{u\to\infty} g(u) = 0$ and

$$\lim_{u\to\infty} \frac{q(u)}{g(u)} = 0,$$

let

$$\delta_b = \{kbg(u), k = 0, 1, 2, ..., [h/bg]\}, \tag{17}$$

be a sparse grid (with respect to the Pickands' grid). Let

$$u' = \left(\frac{q(u)}{g(u)}\right)^{1/c} u, \tag{18}$$

so that as $u \to \infty$, $u' = o(u) \to \infty$. Further assume that $g(u)$ is such that the slowly varying function L in the representation $1 - F(x) = x^{-c}L(x)$ satisfies the condition

$$\lim_{u\to\infty} \frac{L(u')}{L(u)} = 1.$$

Assume that there exists variables $\{\zeta_{b,y}(k)\}_{k=1}^{\infty}$ such that for any $y > 0$ and for any N,

$$(\frac{1}{u'}X(bg), ..., \frac{1}{u'}X(Nbg)|\frac{1}{u'}X(0) > y) \to^D (\zeta_{b,y}(1), ..., \zeta_{b,y}(N)).$$

Theorem 3 : *(Joint convergence of maxima over Pickands' and sparse grids)*

For any Pickands' grid δ_a and sparse grid δ_b defined in (17) and for any $x > 0$, $y > 0$,

1.
$$\lim_{u\to\infty}\frac{q(u)}{1-F(u)}P(M_{\delta_a}(h)\ge uy, M_{\delta_b}(h)\ge u'x)=0.$$

2.
$$\lim_{u\to\infty}\frac{q(u)}{1-F(u)}P(M(h)>uy\cup M_{\delta_b}(h)>u'x)=hy^{-c}H_y(y)+hx^{-c}H'_x(x),$$

where,$0<H'_x(x)<\infty$ is the limit

$$H'_x(x)=\lim_{b\to 0}\frac{1}{b}P(\max_{k\ge 1}\zeta_{b,x}(k)\le x),$$

and $H_y(y)$ is given in (12).

2.2 Increasing intervals

Let

$$M(T)=\max_{t\in[0,T]}X(t),$$
$$M_\delta(T)=\max_{0\le jaq\le T}X(jaq),$$

and u_T be chosen such that as $T\to\infty$,

$$\frac{T}{q(u_T)}(1-F(u_T))=1.$$

For simplicity in notation, let $q=q(u_T)$.

Assume that

1. *Condition* $\Delta(u_{T,1}(x_1),u_{T,2}(x_2))$

   For $0<s<t<T$ and x_i, $i=1,2$ write

   $$\Im^T_{s,t}(x_1,x_2)=\sigma\{(X(v)\le u_{T,j}(x_i):x_i>0,s\le v\le t,i=1,2,j=1,2\},$$

   the sigma field generated by the respective events and

   $$\begin{aligned}\alpha_{T,l}(x_1,x_2) &= \sup\{|P(AB)-P(A)P(B)|:\\ A &\in \Im^T_{0,s}(x_1,x_2),B\in\Im^T_{s+l,t}(x_1,x_2),s\ge 0,l+s\le T\}.\end{aligned}$$

   $\Delta(u_{T,1}(x_1),u_{T,2}(x_2)$ is said to hold for the process $X(t)$ and the family of pair of constants $\{u_{T,1}(x_1),u_{T,2}(x_2)\}$, if $\alpha_{T,l}(x_1,x_2)\to 0$, as $T\to\infty$ for some $l_T=o(T)$. Note that this is a variation of the usual $D(u_n)$ condition, adapted to events generated by two different normalization. see Mladenovic and Piterbarg(2004) for a similar condition.

2. Assume that the X-process satisfies the *No clusters of clusters condition* of Albin(190): This condition is said to hold for $X(t)$ with respect to the grid $\delta = \{jaq(u), j = 0, 1, 2, ...\}$ if for any finite $h > 0$

$$\limsup_{u\to\infty} \frac{1}{1 - F(u)} \sum_{\frac{1}{2}h<jaq<\epsilon T} P(X(0) > u, X(jaq) > u) \to 0, \tag{19}$$

as $\epsilon \to 0$.

Theorem 4: *Maxima over increasing intervals*

Assume that the X-process satisfies the conditions of the previous section as well as the conditions $\Delta(u_T x, u_T y)$ and (19). Then

1. For any Pickands' rrids δ_b given in Theorem 2,

$$\lim_{T\to\infty} P(M(T) \le u_T x, M_{\delta_b}(T) \le u_T y) = \exp[-z^{-c} H_z(b, x, y)].$$

2. For any sparse grid δ_b defined as in Theorem 4, and

$$u'_T = \left(\frac{q(u_T)}{g(u_T)}\right)^{1/c} u_T,$$

assume that the process satisfies the $\Delta(u_T x, u'_T y)$ condition as well as the no clusters of clusters condition given by (19). Then

$$\lim_{T\to\infty} P(M(T) \le u_T x, M_{\delta_b}(T) \le u'_T y) = \exp[-x^{-c} H_x(x) - y^{-c} H'_y(y)]. \tag{20}$$

It is possible to extend Corollary 1 to joint convergence: If δ_b is a Pickands' grid given in (1) of Theorem 2, then under the alternative conditioning (16)

Corollary 2

$$\lim_{T\to\infty} P(M(T) \le ux, M_{\delta_b}(T) \le uy) = \exp[-z^{-c} \hat{H}_z(b, x, y)],$$

where,

$$\hat{H}_z(x, y) = \lim_{a\to 0} \frac{1}{a} \int_1^\infty P(\max_{k\ge 1} \zeta_{a,zw}(k) \le x, \max_{k\ge 1} \zeta_{b,zw}(k) \le y) c w^{-(c+1)} dw.$$

Further,

$$\lim_{b\to 0} \hat{H}_z(b, x, y) = \hat{H}_z(z). \tag{21}$$

For ease of notation, we have given the results for distributions in the domain of attraction of Frechét. However, with some standard changes, it is possible to extend the results to other domains of attraction, See for example Albin(1987,1990) for conditions and proofs of marginal convergence.

It is difficult to verify the conditions and the specific expressions given for $H_x(x)$ and $H_z(b, x, y)$ for processes other than Gaussian processes. However, there are some processes which are transformations of Gaussian processes such as the Rayleigh process for which these conditions may be verified and the specific expressions may be calculated. see Albin(1990) for details. Here we give another example for which it is possible to obtain specific results.

3 Periodogram

Let $\{X_t\}_{t=1}^n$ be a stationary time series with 0 mean and finite variance. The periodogram

$$\begin{aligned} I_n(\omega) = \quad &= \frac{2}{n}|\sum_{t=1}^{n} X_t e^{i\omega t}|^2 \\ &= X_n^2(\omega) + Y_n^2(\omega) \quad , \quad \omega \in [0, \pi] \end{aligned} \tag{22}$$

where

$$X_n(\omega) = \sqrt{2/n}\sum_{t=1}^{n} X_t \cos(\omega t), \tag{23}$$

$$Y_n(\omega) = \sqrt{2/n}\sum_{t=1}^{n} Y_t \sin(\omega t), \tag{24}$$

appears to be the natural estimator of the spectral density function $h(\omega)$, yet it is inconsistent and its erratic behaviour is well known. One reason for this erratic behaviour is that the maximum of the periodogram over any finite interval diverges as $n \to \infty$ almost surely in the order of $2 \log n$. (See for example, Turkman and Walker, 1990) Fundamental reason for this erratic behaviour is that the correlation functions $r_{n,X}(t)$ and $r_{n,Y}(t)$ of the processes $X_n(\omega)$ and $Y_n(\omega)$ having the behaviour

$$1 - \frac{n^2}{3}t + o(t^2),$$

as $t \to 0$. Thus, these processes have second spectral moments which diverge, as $n \to \infty$.

The periodogram plays an important role in tests of hypotheses regarding the jumps in the spectral distribution function. In particular, the maximum of periodogram ordinates over the fourier frequencies $\omega_j = 2\pi j/n$, $j = 1, 2, ..., [\frac{1}{2}(n-1)]$ given by

$$M_{n,I} = \max_{1 \le j \le [\frac{1}{2}(n-1)]} I_n(\omega_j),$$

plays a central role in these tests of hypotheses. The convenience of using this test statistic is that when X_t is a zero mean Gaussian process, the periodogram ordinates over these fourier frequencies constitute an iid standard exponential sample, and the asymptotic distribution of the test statistics is relatively easy to obtain. On the other hand, when X_t is a zero mean, finite variance non-Gaussian process, these ordinates are neither independent nor uncorrelated, but Davis and Mikosch(1999) show that the asymptotic distribution of $M_{n,I}$ still has a similar behaviour.

In principle, tests on the jumps should be constructed based on the maxima of the periodogram over the continuous range of frequencies

$$M_I = \max_{\omega \in [0,\pi]} I_n(\omega),$$

and it is not very clear how much power, if any, one sparses by using the discrete maxima instead of the continuous maxima while constructing these tests. Walker(1965) remarks that indeed greater power can be achieved by using the continuous maxima, indicating that the discrete maximum over the fourier frequencies may not sufficiently approximate the continuous maximum. In fact, when X_t is a Gaussian sequence, a Pickands' grid for the periodogram can be found using a theorem of Bernstein on trigonometric polynomials (See Zygmund, 1959 or Turkman and Walker, 1984):

Theorem: If a trigonometric polynomial of order n,

$$T(x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

satisfies $|T(x)| \leq M$ for every x and for some constant M, then the derivative $T'(x)$ satisfies $|T'(x) \leq nM$.

Clearly, The periodogram is a trigonometric polynomial of order n. If $\delta_a = \{w_j = jaq(n), j = 0, 1, ...\}$ is a partition of $[0, \pi]$, then for any $\omega \in [\omega_j, \omega_{j+1})$, there exists another $\omega^* \in [\omega_j, \omega_{j+1})$ such that

$$I_n(\omega) = I_n(\omega_j) + (\omega - \omega_j) I_n'(w^*).$$

Denote by $M_{\delta_a, I}$, the maximum of the periodogram over the grid δ_a. Then, from Bernstein's Theorem, almost surely

$$\begin{aligned} M_{\delta_a,I} \leq M_I &\leq M_{\delta_a,I} + \max_{\omega_1} |\omega_1 - \omega_j| \max_{\omega \in [0,\pi]} I_n'(\omega) \\ &\leq M_{\delta_a,I} + \pi a q(n) n M_I. \end{aligned}$$

Almost surely

$$\lim_{n\to\infty} \frac{M_I}{2\log n} = 1,$$

(see Turkman and Walker, 1990), hence as $n \to \infty$,

$$\begin{aligned} 0 \leq M_I - M_{\delta_a,I} & \\ &\leq 2\pi a q(u) n \log n. \end{aligned}$$

Therefore, choosing $q(u) = (n \log n)^{-1}$, we see that almost surely

$$\lim_{a\to 0} \lim_{n\to\infty} M_{n,I} - M_{n,\delta_a,I} = 0.$$

These arguments need to be slightly more precise near $\omega = 0$, but we omit the details which can be found in Turkman and Walker (1984).

Thus the fourier frequencies $\omega_j = 2\pi j/n$, $j = 1, 2, ..., [\frac{1}{2}(n-1)]$ form a sparse grid and the maximum over the fourier frequencies and over a dense grid grow with different rates. In fact, as $n \to \infty$,

$$P(M_{n,I} \leq 2x + \log n - 2\log 2) \to \exp(e^{-x}), \tag{25}$$

whereas,

$$P(M_I \leq 2x + 2\log n + \log\log n - \log 3/\pi) \to \exp(e^{-x}). \qquad (26)$$

Thus, if Λ is a random variable with standard Gumbel distribution, then

$$M_{n,I} =^d 2\Lambda + 2\log n - 2\log 2,$$

whereas

$$M_I =^d 2\Lambda + 2\log n + \log\log n - \log 3/\pi,$$

These results show the degree of deviance of the maximum of the periodogram ordinates at fourier frequencies from the maximum over the continuous range of frequencies, differing in the limit by an order of $\log\log n$.

The limit in (25) is given by Walker(1965), whereas the limit(26) is obtained in Turkman and Walker(1984) by showing that as $n \to \infty$,

$$P(M_{n,I} > u_n) \sim \mu(u_n),$$

where $\mu(u_n)$ is the upcrossing intensity of the high level u_n. However, in order to obtain such a result, it is needed to verify that the second order moment of up-crossings given by

$$E[N_{u_n}(N_{u_n} - 1)]$$

is negligibly small, that is

$$E[N_{u_n}(N_{u_n} - 1)] = o(\mu(u_n)), \qquad (27)$$

where N_{u_n} is the number of upcrossings of the level u_n in the interval $[0, \pi]$. Methods which are employed to obtain such results are specific for Gaussian processes and other processes which are simple transformations of Gaussian processes, such as the periodogram for Gaussian time series. It would be interesting to characterize the joint limiting distribution of the periodogram maximum over the continuous range of frequencies and the periodogram maximum over a Pickands' grid in terms of the random sequence $\{\zeta_{a,x}(k)\}_{k=1}^{\infty}$ given in (10). Albin(1990) shows that under the condition (27),

$$\lim_{a\to 0} \frac{1}{a} P(\zeta_{a,x}(1) \leq x) = \lim_{a\to 0} \frac{1}{a} P(\sup_{k\geq 1} \zeta_{a,x} \leq x),$$

therefore the sequence $\{\zeta_{a,x}(k)\}_{k=1}^{\infty}$ must be degenerate in some sense, facilitating part of the tedious technical work.

Characterizations of section 2 give a very detailed and accurate description of the asymptotic relationship between the discrete and continuous maxima, but they have very limited practical use, since the conditions are generally hard to verify and little is known on possible estimators for expressions, such as $H_z(b, x, y)$, defining the degree of dependence between the continuous and discrete extremes. Therefore, it is very important to get simpler and more robust representations for the relationship between continuous and discrete maxima, such as the adjustment (3) suggested by Anderson(2003), which are more adapted for statistical inference, permitting numerical computations and applications. For statistical applications, most interesting case is the joint distribution of maxima over the continuous range and a sparse grid. However, asymptotic results are not particularly useful, as these maxima are asymptotically independent. Therefore, more refined class of models describing the tails of asymptotically independent distributions at sub-asymptotic levels are needed. The study of rates of convergence related to the reported asymptotic results may also be very useful in getting sharper bounds for the adjustment given in (3).

References

Albin, P. (1987) On Extremal theory for non differentiable stationary processes. P.hD. Theses, Dept of Mathematical Statistics, University of Lund

Albin, P. (1990) On extremal theory for stationary processes. *The Ann. of Probab.* 18, 92-128.

Albin, P. (2001) On extremes and streams of upcrossings. *Stochastic processes and their applications*, **94**, 271-300.

Anderson,C.W. (2003) A note on Continuous-time extremes from discrete time observations. *Unpublished research report.*

Berman, S. (1982) Sojourns and extremes of stationary processes. *Ann. Probab.* 10, 1-46.

Hsing, T. and M.R. Leadbetter (1998) on the excursion Random Measure of stationary Processes. *The annals of Probab.*, Vol. 26, 710-742.

Husler, J.(1999) Extremes of a Gaussian Process and the constant H_α, *Extremes 2*, 59-70.

Husler, J. (2004) Dependence between extreme values of discrete and continuous time locally stationary gaussian processes. *Extremes*, **7** 179-190.

Husler, J., ladneva,A., and V. Piterbarg (2011) On clusters of high extremes of Gaussian stationary processes with ϵ-separation. *E.J.Th.Probab.*

Leadbetter,M.R. and H. Rootzen (1983) Extreme value theory for continuous parameter stationary processes. *Z. Wahrsch. verw. Gebiete*, 60, 1-20.

Mladenovic, P and V. Piterbarg(2006) On asymptotic distribution of maxima of complete and incomplete samples from stationary sequences. *Stochastic Processes and their applications* **116** 1977-1991

Piterbarg,V.(2004) Discrete and Continuous time extremes of Gaussian Processes. *Extremes* 7, 161-177.

Robinson, M.E. and Tawn,J.(2000) Extremal analysis of processes sampled at different frequencies. *J.Roy. Statist. Soc. B* 62, 117-35.

Samorodnitsky a, G. and Taqqu, M. (1994) Stable Non-Gaussian Random Processes. Chapman and Hall.

Scotto,M.G, Turkman, K.F. and C.W. Anderson(2003) Extremes of some subsampled time series. *J. Time Ser. Anal.* 24 579-590.

Turkman, K.F. (2011) Continuous and discrete time extremes of stationary processes. *Research report, CEAUL*

Turkman, K.F. and A.M. Walker(1984) On the asymptotic distributions of maxima of trigonometric polynomials with random coefficients. *Adv. in Appl. Probab.* **16** 819-842

Turkman, K.F. and A.M. Walker(1990) A stability result for the periodogram. *Ann. Prob.* **18** 1765-1783

Walker,A.M. (1965) Some asymptotic results for the periodogram of a stationary time series. *J. Austral. Math. Soc.* 5, 107-128.

Zygmund, A.(1959) *Trigonometric series* Vol. 2, Cambridge University Press.