Stuttering Cantor-Like Random Sets

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Abstract. *In each step of the construction of the Cantor set we consider two complementary operations: in the first stage (damage) the middle step of each remaining segment is deleted; in the second stage (random repair) an uniform random segment is united to what remains after deletion. We compute the Hausdorff dimension of the limiting fractal obtained as the intersection of the sets obtained in the* ad infinitum *repetition of this stammering iterative procedure, which as expected is bigger than the Hausdorff dimension of the classical middle Cantor set with no repair.*
Stuttering random Cantor sets are obtained using deletion of uniform random segments both in the damage and in the repair stages in each step of the iterative procedure. The use of general beta random segments in the stuttering construction of Cantor-like random sets is also discussed.

Keywords. Random Cantor sets, Hausdorff dimension, uniform distribution, beta distribution, order statistics.

1. Introduction

Maintenance and repair play an important role in biological and industrial complex systems. Repair random models brought in substantial progress in Probability Theory, cf. v.g. the chapters "Doubling with Repair", and "Mathematical Theory of Reliability Growth" in Gnedenko and Korolev (1996) in the development of the theory of random sums, namely in the case of randomly stopped sums with geometric subordinator.

It is well-known that complex biological systems survive due to redundancy. In complex industrial systems, redundancy in repair is economically sound. Two examples: in the recovery of brain lesions, several regions can contribute to perform, at least partially, the functions of the damaged region; in the recovery of information in damaged hardware, information is extracted both from the damaged and the undamaged tracks. In the next step redundancy is cleared up — an operation that mirrors the mathematical operation of set union, in the sense that it discards repetitions.

We shall first consider step by step attempts to repair the classical middle Cantor set: at each step, deletion of the middle set is followed by a random repair, i.e. a random segment is united to what remains after deletion. As expected, the Hausdorff dimension of this "stuttering" Cantor set is slightly greater than the Hausdorff dimension of the classical Cantor set.

Next we consider a(n uniform) random Cantor set, whose construction "stammers" at each step: starting from $F_0 = [0,1]$, a random segment obtained using two uniform random points as its endpoints is removed, but this is followed by a tentative random repair, in the sense that a random segment, independent from the former one, is united to what remains after the deletion. The procedure is then iteratively repeated in each segment remaining after completion of the previous set.

Random Cantor-like sets (Pesin and Weiss, 1994) can be built up using many interesting randomization procedures.

Aleixo *et al.* (2008, 2009, 2010), Pestana *et al.* (2009) and Rocha *et al.* (2011) exploited systematically the use of beta randomness in the generation of Cantor-like sets. We generalize the uniform stuttering random Cantor set in the framework of beta-randomness.

In what concerns deterministic fractals, as a consequence of Banach's contractive mapping fixed point theorem, under the open set condition on the sequence of similitudes $\psi_i, i = 1, \dots, m$ — i.e. a composition of an isometry and a dilation around some point, whose magnitude is the contraction constants r_i of the similitude — the unique fixed point of ψ is a set whose Hausdorff dimension is s where s is the unique solution of $\sum_{i=1}^{m} r_i^s = 1$ (for details, Falconer (1990) is a readable intermediate level source).

The extension for random fractals is done in terms of expectations.

Similitude has of course to be envisaged in the context of randomness, i.e. the random variables R_i describing the contractions in each step must be equal in distribution.

In such setting (once again, cf. Falconer (1990) for details), the Hausdorff dimension almost surely degenerates in a constant s which is the unique solution of $\sum_{i=1}^{m} \mathbb{E}(R_i^s) = 1$.

In what follows we shall have occasion to use this result, in the special framework where the the random contractions are equal in distribution to the random sets resulting from the first step of the iterative construction of the random fractal. This is a simple consequence of taking $[0,1]$ as the initial set, and hence — using, for illustrative purposes, the notation we shall use in describing the Cantor set with deterministic damage and random repair, $\frac{\widetilde{S}_{\bullet,k}}{\widetilde{S}_{\bullet,k-1}} \stackrel{\mathrm{d}}{=} \frac{\widetilde{S}_{\bullet,1}}{1} = \widetilde{S}_{\bullet,1}$.

On the other hand, the number of the contraction random variables acting at each step is a random variables, and the useful tool to compute expectations in such circumstances is of course conditioning and then using the total (or averaging) probability theorem. In other words, the simple way of computing expectations is to perform the computation in two steps, first computing conditional expectations, and then averaging them (Ross, 2009).

2. The stuttering Cantor set with random repair

Let $C_0 = [0,1]$.

- Construction of C_1:
  1. Delete the middle set $\left(\frac{1}{3}, \frac{2}{3}\right)$.
  2. Generate two random points (in the usual sense of uniformly distributed) Y_1 and Y_2 in $[0,1]$.

     $$C_1 = \left[C_0 - \left(\frac{1}{3}, \frac{2}{3}\right)\right] \cup (Y_{1:2}, Y_{2:2}),$$

     where as usual $Y_{k:n}$ denotes the k-th order statistic in a random sample of size n.
- C_k results from applying the above "stuttering", procedure to each segment $S_{i,k-1}$ of C_{k-1}, $k = 2, 3, \dots$:
  1. Delete the middle set $M_{i,k-1}$ from each segment $S_{i,k-1}$ in C_{k-1};
  2. Generate two random points $Y_{1;i,k-1}$ and $Y_{2;i,k-1}$ in $S_{i,k-1}$, and rename $Y_{1:2;i,k-1}$ and $Y_{2:2;i,k-1}$ its minimum and its maximum, respectively;
  3. C_k is what results from this "stuttering random repair":

     $$C_k = \bigcup_i \{[S_{i,k-1} - M_{i,k-1}] \cup \cup (Y_{1:2;i,k-1}, Y_{2:2;i,k-1})\}.$$
- $\mathcal{C} = \bigcap_{k=1}^{\infty} C_k$.

The random set C_1 is a mixture of

$$N_1 = \begin{cases} 1 & 2 & 3 \\ \frac{2}{9} & \frac{2}{3} & \frac{1}{9} \end{cases}$$

random variables (some of which are degenerate):

$$C_1=\bigcup_{i=1}^{N_1} S_{1,i}=\begin{cases}[0,1] \\ \quad \text{with probability } \frac{2}{9} \\ \left[0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right] \\ \quad \text{with probability } \frac{2}{9} \\ [0,Y_{2:2}]\cup\left[\frac{2}{3},1\right] \\ \quad \text{with probability } \frac{2}{9} \\ \left[0,\frac{1}{3}\right]\cup[Y_{1:2},1] \\ \quad \text{with probability } \frac{2}{9} \\ \left[0,\frac{1}{3}\right]\cup[Y_{1:2},Y_{2:2}]\cup\left[\frac{2}{3},1\right] \\ \quad \text{with probability } \frac{1}{9}\end{cases}$$

and C_k is the union of $N_k=\sum_{j=1}^{k} N_{1,j}$ of independent replica of the random variable N_1. Hence C_k is the union of a random number of mixtures of random variables $S_{k,i}$ “similar” (up to scaling) to $S_{1,i}$ as exhibited above,

$$\bigcup_i\{[S_{i,k-1}-M_{i,k-1}]\cup(Y_{1:2;\,i,k-1},Y_{2:2;\,i,k-1})\}=\bigcup_{i=1}^{N_k} S_{k,i};$$

in what follows we denote $\widetilde{S}_{k,i}$ the random length of the segment $S_{k,i}$.

Hence, self-similarity exists at all steps, and the Hausdorff dimension of $\mathcal{C}$ is the value s which is the solution of $\sum_{i=1}^{N_1}\mathbb{E}\left[\widetilde{S}_{1,i}^s\right]=1$, as discussed at the end of Section 1.

From the useful rule

$$\mathbb{E}_T\,[T]=\mathbb{E}_W\left[\mathbb{E}_{T|W}\,(T\mid W)\right]$$

to compute expectations in a conditionally hierarchical or mixing framework, as $Y_{1:2}\frown Beta(1,2)$, $Y_{2:2}\frown Beta(2,1)$ — and hence, from the general result $X\frown Beta(p,q)\Longrightarrow 1-X\frown Beta(q,p)$, $1-Y_{1:2}\frown Beta(2,1)$ —, and as the probability density function of

$$Y_{2:2}\,|_{0<Y_{1:2}<\frac{1}{3}<Y_{2:2}<\frac{2}{3}}\overset{\mathrm{d}}{=}1-Y_{1:2}\,|_{\frac{1}{3}<Y_{1:2}<\frac{2}{3}<Y_{2:2}<1}$$

is $f(y)=3\,\mathbb{I}_{(\frac{1}{3},\frac{2}{3})}(y)$, and the probability density function of $Y_{2:2}-Y_{1:2}\,|_{Y_{1:2}>\frac{1}{3}\wedge Y_{2:2}<\frac{2}{3}}$ is

$$f_{Y_{2:2}-Y_{1:2}|_{Y_{1:2}>\frac{1}{3}\wedge Y_{2:2}<\frac{2}{3}}}(x)=18\left(\frac{1}{3}-x\right)\mathbb{I}_{(0,\frac{1}{3})}(x),$$

and therefore $\mathbb{E}\left[\left(Y_{2:2}\,|_{0<Y_{1:2}<\frac{1}{3}<Y_{2:2}<\frac{2}{3}}\right)^s\right]=\mathbb{E}\left[\left(1-Y_{1:2}\,|_{\frac{1}{3}<Y_{1:2}<\frac{2}{3}<Y_{2:2}<1}\right)^s\right]=$

$$\frac{3\,\left((\frac{2}{3})^{s+1}-(\frac{1}{3})^{s+1}\right)}{s+1}$$

and

$$\mathbb{E}\left[\left(Y_{2:2}-Y_{1:2}\,|_{Y_{1:2}>\frac{1}{3}\wedge Y_{2:2}<\frac{2}{3}}\right)^s\right]=\frac{2}{3^s\,(s+1)(s+2)}\,,$$

it follows that s is the solution of

$$\frac{2}{9}+\frac{10}{3^{s+2}}+\frac{4\,\left((\frac{2}{3})^{s+1}-(\frac{1}{3})^{s+1}\right)}{3\,(s+1)}+$$

$$+\frac{2}{3^{s+2}\,(s+1)(s+2)}=1,$$

i.e. $s\approx 0.741122$, which as expected is bigger than $\frac{\ln(2)}{\ln(3}$, the Hausdorff dimension of the deterministic middle Cantor set.

3. Stuttering Cantor-like random sets

We now go a step forward in the randomization of the Cantor set: instead of the middle third of each segment, in the damage phase of each step a random (uniform) segment is deleted, and in the repair stage an independent random segment is added. Formally:

Let $F_0=[0,1]$.

- Construction of F_1:
    1. (Damaging stage) Generate two uniform random points X_1 and X_2 in $[0,1]$ and delete from $[0,1]$ the set $(X_{1:2},X_{2:2})$.
    2. (Repair stage) Generate two uniform random points Y_1 and Y_2 in $[0,1]$, independent from (X_1,X_2)

    $$F_1=[F_0-(X_{1:2},X_{2:2})]\cup(Y_{1:2},Y_{2:2})\,.$$

  As $(X_1,X_2,,Y_1,Y_2)\overset{\mathrm{d}}{=}(U_1,U_2,U_3,U_4)$ where the U_k, $k=1,\ldots,4$ are independent replica of the standard uniform random variable, and there are $4!=24$ possible reorderings to consider when dealing with order statistics, it is easily established that random set F_1 is a mixture of

$$\widetilde{N}_1 = \left\{ \begin{array}{ccc} 1 & 2 & 3 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array} \right.$$

random variables (some of which are degenerate): $F_1 = \bigcup_{i=1}^{N_1} S_{1,i}$, or more explicitly

$$F_1 = \begin{cases} [0,1] \quad (\mathbf{a}) \\ \quad \text{if } Y_{1:2} < X_{1:2} \text{ and } Y_{2:2} > X_{2:2}, \\ \quad \text{i.e. with probability } \frac{1}{6} \\ [0,X_{1:2}] \cup [X_{2:2},1] \quad (\mathbf{b}) \\ \quad \text{if } Y_{2:2} < X_{1:2} \text{ or } Y_{1:2} > X_{2:2}, \\ \quad \text{i.e. with probability } \frac{1}{3} \\ [0,Y_{2:2}] \cup [X_{2:2},1] \quad (\mathbf{c}) \\ \quad \text{if } Y_{1:2} < X_{1:2} < Y_{2:2} < X_{2:2}, \\ \quad \text{with probability } \frac{1}{6} \\ [0,X_{1:2}] \cup [Y_{1:2},1] \quad (\mathbf{d}) \\ \quad \text{if } X_{1:2} < Y_{1:2} < X_{2:2} < Y_{2:2}, \\ \quad \text{with probability } \frac{1}{6} \\ [0,X_{1:2}] \cup [Y_{1:2},Y_{2:2}] \cup [X_{2:2},1] \quad (\mathbf{e}) \\ \quad \text{if } X_{1:2} < Y_{1:2} < Y_{2:2} < X_{2:2}, \\ \quad \text{with probability } \frac{1}{6}\,. \end{cases}$$

- F_k results from applying the above "stuttering" procedure to each segment $S_{i,k-1}$ of F_{k-1}, $k = 2,3,\ldots$: in each segment $S_{i,k-1}$ in F_{k-1}

  1. (Damaging stage) Generate two random points $X_{1;i,k-1}, X_{2;i,k-1}$, and delete the middle random segment $(X_{1:2;i,k-1}, X_{2:2:i,k-1})$ from the corresponding segment $S_{i,k-1}$ in F_{k-1}.
  2. (Repair stage) Generate two random points $Y_{1;i,k-1}$, $Y_{2;i,k-1}$, independent from $(X_{1;i,k-1}, X_{2;i,k-1})$, and perform the union of the "damaged set" obtained in the previous stage with $(Y_{1:2;i,k-1}, Y_{2:2;i,k-1})$.
  3. F_k is what results from this "stuttering random repair":

$$\bigcup_i \{[S_{i,k-1} - (X_{1:2;i,k-1}, X_{2:2:i,k-1})] \cup$$

$$\cup \, (Y_{1:2;i,k-1}, Y_{2:2;i,k-1})\}\,.$$

- $\mathcal{F} = \bigcap_{k=1}^{\infty} F_k$ is the stuttering Cantor-like random set.

F_k is the union of $\widetilde{N}_k = \sum_{j=1}^{\widetilde{N}_{k-1}} \widetilde{N}_{1,j}$ of independent replica of the random variable $\widetilde{N}_1$. Hence F_k is the union of a random number of mixtures of random variables $F_{k,i}$ "similar" (up to scaling) to F_1 as exhibited above,

$$\bigcup_i \{[S_{i,k-1} - (X_{1:2;i,k-1}, X_{2:2:i,k-1})] \cup$$

$$\cup \, (Y_{1:2;i,k-1}, Y_{2:2;i,k-1})\} = \bigcup_{i=1}^{\widetilde{N}_k} S_{k,i}.$$

Observe also that in terms of the independent replica $U_k \frown Beta(1,1)$, $k = 1,\ldots,4$, the expressions (**b**), (**c**), (**d**) and (**e**) are

$$(\mathbf{b}) = \{[0,U_{3:4}] \cup [U_{4:4},1]\} \cup \{[0,U_{1:4}] \cup [U_{2:4},1]\}$$

$$(\mathbf{c}) = [0,U_{3:4}] \cup [U_{4:4},1]$$

$$(\mathbf{d}) = [0,U_{1:4}] \cup [U_{2:4},1]$$

$$(\mathbf{e}) = [0,U_{1:4}] \cup [U_{2:4},U_{3:4}] \cup [U_{4:4},1]\,,$$

a useful representation when the goal is to compute expectations.

In fact, observing that

$$U_{3:4} \stackrel{\mathrm{d}}{=} 1 - U_{2:4} \frown Beta(3,2)$$

and that

$$1 - U_{4:4} \stackrel{\mathrm{d}}{=} U_{1:4} \stackrel{\mathrm{d}}{=} U_{3:4} - U_{2:4} \frown Beta(1,4),$$

using the methodology already explained in Section 1 and applied in Section 2, the Hausdorff dimension s of the fractal $\mathcal{F}$ is the solution of

$$\frac{1}{6} + \frac{8}{(s+3)(s+4)} + \frac{28}{(s+1)(s+2)(s+3)(s+4)} = 1,$$

i.e., $s \approx 0.669783$.

In Pestana *et al.* (2009) it has been shown that the Hausdorff dimension of a

Cantor-like random fractal obtained by the iterative removal of a middle random set (in the precise sense that its endpoints were the order statistics of two uniform points in the segments of the previous iteration) — hence with no repair stage — is approximately 0.56155.

As expected, the limiting random fractal obtained with a repair stage is more "dense" in the linear initial segment [0,1] than the random fractal with no repair; observe that in fact it is more dense that the deterministic Cantor fractal. On the other hand, the limiting Cantor fractal with random repair, with Hausdorff dimension approximately 0.741122, is of course much denser.

4. Stuttering Beta(p,q) Cantor-like random sets

As the standard uniform random variable is a $Beta(1,1)$, we can now investigate more general settings, generating the random points with $Beta(p,q)$ distribution (or even $Beta(p,q)$ in the damage stage and $Beta(\tilde{p},\tilde{q})$ in the repair stage).

Just to illustrate what happens, we generate random points with $Beta(2,2)$ distribution to use in both, the damage and the repair stage.

In this case, the probabilities associated with the set C_1, in the construction of the stuttering Cantor set with random repair, are given by

- $2\times\left(\frac{7}{27}\right)^2 \approx 13,5\%$ for $C_1=[0,1]$ or $C_1=\left[0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right]$;
- $\frac{2\times 7\times 13}{27^2}\approx 25\%$ for $C_1=[0,Y_{2:2}]\cup\left[\frac{2}{3},1\right]$ or $C_1=\left[0,\frac{1}{3}\right]\cup[Y_{1:2},1]$;
- $\left(\frac{13}{27}\right)^2\approx 23\%$ for $C_1=\left[0,\frac{1}{3}\right]\cup[Y_{1:2},Y_{2:2}]\cup\left[\frac{2}{3},1\right]$.

The probability density function of

$$Y_{2:2}\,|_{0<Y_{1:2}<\frac{1}{3}<Y_{2:2}<\frac{2}{3}} \overset{\mathrm{d}}{=} 1-Y_{1:2}\,|_{\frac{1}{3}<Y_{1:2}<\frac{2}{3}<Y_{2:2}<1}$$

is

$$f(y)=\frac{162}{13}\left(y-y^2\right)\mathbb{I}_{\left(\frac{1}{3},\frac{2}{3}\right)}(y),$$

and the probability density function of

$$Y_{2:2}-Y_{1:2}\,|_{Y_{1:2}>\frac{1}{3}\wedge Y_{2:2}<\frac{2}{3}}$$

is

$$f_{Y_{2:2}-Y_{1:2}|_{Y_{1:2}>\frac{1}{3}\wedge Y_{2:2}<\frac{2}{3}}}(x) = 72\left(\tfrac{27}{13}\right)^2\left(\tfrac{47}{2430}-\tfrac{4x}{81}-\tfrac{13x^2}{162}+\tfrac{x^3}{6}-\tfrac{x^5}{30}\right)\mathbb{I}_{(0,\frac{1}{3})}(x),$$

and therefore

$$\begin{aligned}&\mathbb{E}\left[\left(Y_{2:2}\,|_{0<Y_{1:2}<\frac{1}{3}<Y_{2:2}<\frac{2}{3}}\right)^s\right]\\ =\;&\mathbb{E}\left[\left(1-Y_{1:2}\,|_{\frac{1}{3}<Y_{1:2}<\frac{2}{3}<Y_{2:2}<1}\right)^s\right]\\ =\;&\frac{2\left(-7-2s+2^{s+2}(s+5)\right)}{13(s+2)(s+3)3^{s-1}}\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\left[\left(Y_{2:2}-Y_{1:2}\,|_{Y_{1:2}>\frac{1}{3}\wedge Y_{2:2}<\frac{2}{3}}\right)^s\right]\\ =\;&\frac{8(338+s(249+4s(s+14)))}{169(s+1)(s+2)(s+3)(s+4)(s+6)3^{s-2}}\end{aligned}$$

it follows that s is the solution of

$$\frac{2}{27^2}\left(49+\frac{449}{3^s}+\frac{28\left(-7-2s+2^{s+2}(s+5)\right)}{(s+2)(s+3)3^{s-1}}\right.\\ \left.+\frac{4(338+s(249+4s(s+14)))}{(s+1)(s+2)(s+3)(s+4)(s+6)3^{s-2}}\right)=1,$$

i.e. $s\approx 0.764267$, which is slightly bigger than the respective Hausdorff dimension of the stuttering Cantor set with random uniform repair, and as expected is bigger than $\frac{\ln(2)}{\ln(3}$, the Hausdorff dimension of the deterministic middle Cantor set.

In what concerns the correspondent stuttering $Beta(2,2)$ Cantor-like random set, considering that

$$\begin{aligned}&\mathbb{E}\left[U_{1:4}^s\right]=\mathbb{E}\left[(1-U_{4:4})^s\right]\\ =\;&120960\left(\frac{\Gamma[s+2]}{\Gamma[s+10]}+\frac{6\Gamma[s+3]}{\Gamma[s+11]}+\frac{12\Gamma[s+4]}{\Gamma[s+12]}\right.\\ &\left.+\frac{8\Gamma[s+5]}{\Gamma[s+13]}\right),\end{aligned}$$

$$\begin{aligned}&\mathbb{E}\left[U_{3:4}^s\right]=\mathbb{E}\left[(1-U_{2:4})^s\right]\\ =\;&432\left(\frac{9\Gamma[s+6]}{\Gamma[s+10]}+\frac{6\Gamma[s+7]}{\Gamma[s+11]}-\frac{20\Gamma[s+8]}{\Gamma[s+12]}\right.\\ &\left.+\frac{8\Gamma[s+9]}{\Gamma[s+13]}\right),\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\left[(U_{3:4}-U_{2:4})^s\right]\\ =\;&\frac{120960}{385}\left(\frac{86\Gamma[s+1]}{\Gamma[s+9]}+5\left(\frac{61\Gamma[s+2]}{\Gamma[s+10]}\right.\right.\\ &\left.\left.+\frac{68\Gamma[s+3]}{\Gamma[s+11]}+\frac{28\Gamma[s+4]}{\Gamma[s+12]}+\frac{4\Gamma[s+5]}{\Gamma[s+13]}\right)\right),\end{aligned}$$

it follows that s is the solution of

$$\frac{1}{6}\left(1+6\mathbb{E}\left[U_{1:4}^s\right]+4\mathbb{E}\left[U_{3:4}^s\right]+\mathbb{E}\left[(U_{3:4}-U_{2:4})^s\right]\right)=1,$$

i.e. $s \approx 0.751592$. This value is slightly smaller than the Hausdorff dimension of the stuttering $Beta(2,2)$ Cantor set with random repair calculated above. It is interesting to compare the value of the Hausdorff dimension for different kinds of Cantor sets, constructed by several combinations of damage and repair methodologies, see table 1.

Table 1: Hausdorff dimension

Damage \ Repair	B_1	B_2	B_3
A_1	0.630930	0.741122	0.764267
A_2	0.561553	0.669783	—
A_3	0.666305	—	—
A_4	0.618907	—	0.751592

A_1 - Deterministic, uniform (Middle third damage segment),
A_2 - Random, uniform (the extremes of the damage segment are the minimum and maximum of an observed two dimension uniform sample),
A_3 - Deterministic, $Beta(2,2)$ (the extremes of the damage segment are the expected values of the minimum and maximum of an two dimension $Beta(2,2)$ random sample),
A_4 - Random, $Beta(2,2)$ (the extremes of the damage segment are the minimum and maximum of an observed two dimension $Beta(2,2)$ sample),

B_1 – Nihil repair segment,
B_2 – Random, uniform (the extremes of the repair segment are the minimum and maximum of an observed two dimension uniform sample),
B_3 – Random, $Beta(2,2)$ (the extremes of the repair segment are the minimum and the maximum of an observed two dimension $Beta(2,2)$ sample).

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References

[1] Aleixo, S.M., Rocha, J.L. and Pestana, D.D. Populational growth models proportional to beta densities with Allee effect. Am. Inst. Phys. 2008; 1124, 3–12.

[2] Aleixo, S.M., Rocha, J.L. and Pestana, D.D. Dynamical behaviour in the parameter space: new populational growth models proportional to beta densities. In Luzar-Stiffler, V., Jarec, I., Bekic, Z. (eds), Proc. ITI 2009, 31th Int. Conf. Information Technology Interfaces; 213–218.

[3] Aleixo, S.M., Rocha, J.L. and Pestana, D.D. Dynamical analysis in probabilistic models: random Beta(p,2)-Cantor sets, Notas e Comunicações CEAUL. 2010; 6.

[4] Falconer, K. *Fractal Geometry, Mathematical Foundations and Applications.* Chichester: Wiley; 1990.

[5] Gnedenko, B.V., and Korolev, V.Yu. *Random Summation. Limit Theorems and Application*, Boca Raton: CRC; 1996.

[6] Pesin, Y. and Weiss, H. On the dimension of the deterministic and random Cantor-like sets. Math. Res. Lett. 1994;1, 519–529.

[7] Pestana, D.D., Aleixo, S.M. and Rocha, J.L. The Beta(p,1) extensions of the random (uniform) Cantor sets. In Luzar-Stiffler, V., Jarec, I. and Bekic, Z. (eds.), Proc. ITI 2009, 31th Int. Conf. Information Technology Interfaces; 279–284.

[8] Pestana, D.D., Aleixo, S.M. and Rocha, J.L. Hausdorff Dimension of the Random Middle Third Cantor Set. Disc. Math. Prob. Stat. 2009; 29, 119–221.

[9] Rocha, J.L., Aleixo, S.M. and Pestana, D.D. Beta(p,q)-Cantor Sets: Determinism and Randomness. In Chaos Theory: Modeling, Simulation and Applications, Skiadas, C.H., Dimotikalis, Y. and Skiadas, C. (eds), World Scientific Publishing, 2011 (in press).

[10] Ross, S.M. *A First Course in Probability*, London: Prentice Hall, 2009.