# *Tail dependence between order statistics*

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**Abstract:** In this work we introduce the *$s,k$-extremal coefficients* for studying the tail dependence between the $s$-th lower and $k$-th upper order statistics of a random vector. Some general properties are derived for these dependence measures which can be expressed via copulas of random vectors. Its relations with other extremal dependence measures used in the literature are discussed, such as multivariate tail dependence coefficients, the coefficient $\eta$ of tail dependence, coefficients based on tail dependence functions, the extremal coefficient $\epsilon$, the multivariate extremal index and an extremal coefficient for min-stable distributions. Several examples are presented to illustrate the results, including multivariate exponential and multivariate Gumbel distributions widely used in applications.

**Keywords:** Tail dependence, order statistics, measures of tail dependence, multivariate extreme value distribution.

# 1 Introduction

The concepts of tail dependence are standard tools to describe the amount of extremal dependence between random variables. Tail dependence coefficients (upper and lower) measure the probability of occurring extreme values (very large or small) for one random variable (r.v.) given that another assumes an extreme value too. These dependence measures can be expressed via copulas of random vectors which capture those properties of the joint distribution which are scale invariant.
For a random pair $(Z, W)$, the *upper tail dependence coefficient* is given by

$$\lambda_U(W|Z) = \lim_{t\uparrow 1} P(F_W(W) > t|F_Z(Z) > t), \tag{1}$$

where $F_X$ denotes the d.f. of r.v. $X$, and the *lower tail dependence coefficient* is defined as

$$\lambda_L(W|Z) = \lim_{t\downarrow 0} P(F_W(W) \leq t|F_Z(Z) \leq t). \tag{2}$$

If some of these coefficients is positive the r.v.'s $Z$ and $W$ are said to be dependent on the respective tail.

For random pairs with Normal distribution, Sibuya (1960, [25]) presented the interesting result that how high a correlation we choose, if we go far enough into the tail, extreme events appear to occur independently in each margin. Resnick (1983, [20]) extended this result to the $d$-dimensional multivariate Normal distribution. Schmidt (2002, [22]) considered the more general class of elliptical distributions (which includes the multivariate normal and $t$-distributions) and, in contrast to the bivariate normal, the correlation in the bivariate $t$-distribution plays a surprising role. Even for negative and zero correlations, we find asymptotic dependence in the upper tail which increases as the number of degrees of freedom decreases and the marginal distributions become heavier-tailed. Ledford and Tawn (2003, [13]), Zhang and Huang (2006, [31]) extended the definition of tail dependence coefficients to lag-$k$ tail dependence of sequences of random variables with identical marginal distribution. Heffernan *et al.* (2007, [7]) computed these tail coefficients for M4 class of processes introduced in Smith and Weissman (1996, [27]) and Extended M4 class which includes asymptotic independence. Brummelhuis (2008, [2]) characterized the serial dependence in ARCH(1) process as quantified by the lower tail dependence coefficient and some of its generalizations. Ferreira and Canto e Castro (2008, [5]) presented an in-depth study of the serial tail dependence of sequences of levels persisting in time for a fixed period.

The tail dependence coefficients can be related with other dependence measures such as the extremal coefficient $\epsilon$ (Tiago de Oliveira 1962/63, [28]), the coefficient $\eta$ of tail dependence (Ledford and Tawn 1996, [11]; 1997, [12]) and the conditional version of Spearman' s rho $\rho(p)$ in the bivariate setting (Schmid and Schmidt 2007, [23]).

Multivariate formulations for tail dependence coefficients can be used to describe the amount of dependence in the upper/lower orthant tail of a multivariate distribution. Li (2006 [14], 2008 [15]) fully characterizes the tail dependence of multivariate Marshall-Olkin copulas and Ferreira (2008, [4]) the dependence between two multivariate extreme value distributions. Wolff (1980, [30]), Nelsen (1996, [19]) and Schmid and Schmidt (2007, [23]) consider multivariate concepts of tail dependence based on weighting of copulas.

The upper and lower tail dependence coefficients can be generalized to random vectors $\mathbf{Z} = (Z_1, ..., Z_d)$ and $\mathbf{W} = (W_1, ..., W_d)$, with definition (Joe 1997 [8])

$$\lambda_U(\mathbf{W}|\mathbf{Z}) = \lambda_U\big(\min_{i=1,...,d} F_{W_i}(W_i)|\min_{i=1,...,d} F_{Z_i}(Z_i)\big) \tag{3}$$

and

$$\lambda_L(\mathbf{W}|\mathbf{Z}) = \lambda_L\big(\max_{i=1,...,d} F_{W_i}(W_i)|\max_{i=1,...,d} F_{Z_i}(Z_i)\big). \tag{4}$$

We remark that, though the multivariate tail dependence measures may be represented in terms of bivariate coefficients, they have highlighted new aspects of the dependence in vectors.

Results concerning the dependence structure between order statistics have been presented in literature. For instance, Tukey (1958, [29]) has shown that if the r.v.'s $X_i$, $i = 1, ..., d$, are i.i.d. with "subexponential" d.f. in both tails, then the covariance of $X_{i:d}$ and $X_{j:d}$, respectively the $i$-th and $j$-th order statistics, decreases as $i$ and $j$ draw apart (see also Kim and David, 1990 [9]). In the independent and identically distributed case the order statistics have the Markov property implying another type of dependence (Arnold *et al.* 1992 [1] and David 1981 [3], Chap. 2).

Frahm (2006, [6]) considered the upper coefficient $\lambda_U$ for r.v.'s $W = \min_{i=1,...,d} F_{X_i}(X_i)$ and $Z = \max_{i=1,...,d} F_{X_i}(X_i)$, and $\lambda_L$ for $W = \max_{i=1,...,d} F_{X_i}(X_i)$ and $Z = \min_{i=1,...,d} F_{X_i}(X_i)$, where $\mathbf{X} = (X_1, ..., X_d)$ is a random vector. Several properties were deduced for these extremal coefficients and applications were made for elliptical distributions.

Here we present some computation formulas and properties for $\lambda_U$ and $\lambda_L$ when we consider that $W$ and $Z$ are order statistics of a $d$-dimensional random vector $\mathbf{X}$ (Section 2). Some properties have as a particular case the ones derived in Frahm (2006, [6]). We give particular emphasis to the computation of these coefficients in multivariate extreme value distributions (MEV), as well as distributions that are attracted to those (Section 3).

We also relate these coefficients with other known in literature. More precisely, in Section 2, we consider the multivariate tail dependence coefficients of Li (2009, [16]), the coefficient of tail dependence of Ledford and Tawn (1996, [11]; 1997, [12]) extended to a $d$-dimensional framework and coefficients derived from the tail dependence function of Klüppelberg *et al.* (2008, [10]).

Section 3 is devoted to MEV distributions. Here we will state connections with the extremal coefficient (Tiago de Oliveira 1962/63, [28]; Smith 1990, [26]), with the multivariate extremal index (Nandagopalan 1990, [18]), with the spectral measure and with an extremal coefficient for min-stable distributions.

Some examples will illustrate the results. We built some multivariate distributions and consider others of recognized interest for applications as Marshall-Olkin (Section 2) and Gumbel (Section 3).

## 2 Definitions and Properties

Let $\mathbf{X} = (X_1, ..., X_d)$ be a random vector and $X_{1:d} \leq ... \leq X_{d:d}$ the order statistics of $(F_{X_1}(X_1), ..., F_{X_d}(X_d))$. For integers $s$ and $k$ such that $1 \leq s < d-k+1 \leq d$, the *upper* $s, k$*-extremal coefficient* of $\mathbf{X}$ is defined by

$$\lambda_U(X_{s:d}|X_{d-k+1:d}) = \lim_{t\uparrow 1} P(X_{s:d} > t|X_{d-k+1:d} > t), \tag{5}$$

and the *lower $s,k$-extremal coefficient* of $\mathbf{X}$

$$\lambda_L(X_{d-k+1:d}|X_{s:d}) = \lim_{t\downarrow 0} P(X_{d-k+1:d} \le t | X_{s:d} \le t). \tag{6}$$

In engineering, coefficient $\lambda_L(X_{d-k+1:d}|X_{s:d})$ ($\lambda_U(X_{s:d}|X_{d-k+1:d})$) can be interpreted as the limiting probability that the $k$-th best ($s$-th worst) performer in a system is attracted by the $s$-th worst ($k$-th best) one, provided the latter has an extremely bad (good) performance.

In mathematical finance, the value-at-risk at probability level $t$ of a random asset $Z$ is given by the quantile function evaluated at $t$, $F_Z^{-1}(t) = \inf\{x : F_Z(x) \ge t\}$. Therefore, $\lambda_L(X_{d-k+1:d}|X_{s:d})$ can be viewed as the limiting conditional probability that $X_{d-k+1:d}$ violates its value-at-risk at level $t$, given that $X_{s:d}$ has done so. This interpretation holds vice versa regarding the upper coefficient.

We start by stating some formulas to compute these coefficients, based on the copula function $C$ of vector $\mathbf{X}$ and the corresponding survival copula function $\overline{C}$. We recall that, for $(u_1, ..., u_d) \in [0,1]^d$,

$$C(u_1, ..., u_d) = P(F_{X_1}(X_1) \le u_1, ..., F_{X_d}(X_d) \le u_d) \tag{7}$$

and

$$\overline{C}(u_1, ..., u_d) = P(F_{X_1}(X_1) > 1-u_1, ..., F_{X_d}(X_d) > 1-u_d). \tag{8}$$

From now on it is conventioned that $P\big(\cap_{i\in\emptyset} A_i\big) = 1$ for any events $A_i$. Furthermore, we will always denote $F_i$ as the family of all subsets of $\{1, ..., d\}$ with cardinal equal to $i$ and $\overline{I}$ the complement set of $I \in F_i$ in $\{1, ..., d\}$.

**Proposition 2.1** *The $s,k$-extremal coefficients satisfy:*

$$\lambda_U(X_{s:d}|X_{d-k+1:d}) = \lim_{t\uparrow 1} \frac{\displaystyle\sum_{0\le i\le s-1} \sum_{I\in F_i} \sum_{J\subset \overline{I}} (-1)^{|J|} C(t^{\mathbb{1}\{1\in I\cup J\}}, ..., t^{\mathbb{1}\{d\in I\cup J\}})}{1 - \displaystyle\sum_{0\le i\le k-1} \sum_{I\in F_i} \sum_{J\subset I} (-1)^{|J|} C(t^{\mathbb{1}\{1\in \overline{I}\cup J\}}, ..., t^{\mathbb{1}\{d\in \overline{I}\cup J\}})} \tag{9}$$

$$\lambda_U(X_{s:d}|X_{d-k+1:d}) = \lim_{t\uparrow 1} \frac{\displaystyle\sum_{0\le i\le s-1} \sum_{I\in F_i} \sum_{J\subset I} (-1)^{|J|} \overline{C}((1-t)^{\mathbb{1}\{1\in \overline{I}\cup J\}}, ..., (1-t)^{\mathbb{1}\{d\in \overline{I}\cup J\}})}{1 - \displaystyle\sum_{0\le i\le k-1} \sum_{I\in F_i} \sum_{J\subset \overline{I}} (-1)^{|J|} \overline{C}((1-t)^{\mathbb{1}\{1\in I\cup J\}}, ..., (1-t)^{\mathbb{1}\{d\in I\cup J\}})} \tag{10}$$

$$\lambda_L(X_{d-k+1:d}|X_{s:d}) = \lim_{t\downarrow 0} \frac{\displaystyle\sum_{0\le i\le k-1} \sum_{I\in F_i} \sum_{J\subset I} (-1)^{|J|} C(t^{\mathbb{1}\{1\in \overline{I}\cup J\}}, ..., t^{\mathbb{1}\{d\in \overline{I}\cup J\}})}{1 - \displaystyle\sum_{0\le i\le s-1} \sum_{I\in F_i} \sum_{J\subset \overline{I}} (-1)^{|J|} C(t^{\mathbb{1}\{1\in I\cup J\}}, ..., t^{\mathbb{1}\{d\in I\cup J\}})} \tag{11}$$

$$\lambda_L(X_{d-k+1:d}|X_{s:d}) = \lim_{t\downarrow 0} \frac{\displaystyle\sum_{0\le i\le k-1} \sum_{I\in F_i} \sum_{J\subset \overline{I}} (-1)^{|J|} \overline{C}((1-t)^{\mathbb{1}\{1\in I\cup J\}}, ..., (1-t)^{\mathbb{1}\{d\in I\cup J\}})}{1 - \displaystyle\sum_{0\le i\le s-1} \sum_{I\in F_i} \sum_{J\subset I} (-1)^{|J|} \overline{C}((1-t)^{\mathbb{1}\{1\in \overline{I}\cup J\}}, ..., (1-t)^{\mathbb{1}\{d\in \overline{I}\cup J\}})} \tag{12}$$

*where $\mathbb{1}\{\cdot\}$ denotes the indicator function.*

**Proof.** Let $A_d(t) = \sum_{i=1}^d \mathbb{1}_{\{F_{X_i}(X_i) > t\}}$ and $B_d(t) = \sum_{i=1}^d \mathbb{1}_{\{F_{X_i}(X_i) \le t\}} = d - A_d(t)$. We have, $P(X_{s:d} > t) = P(B_d(t) \le s-1)$ and $P(X_{d-k+1:d} > t) = P(A_d(t) \ge k)$. Since

$$\lambda_U(X_{s:d}|X_{d-k+1:d}) = \lim_{t\uparrow 1} \frac{P(X_{s:d} > t)}{P(X_{d-k+1:d} > t)} = \lim_{t\uparrow 1} \frac{\sum_{i=0}^{s-1} P(B_d(t) = i)}{1 - \sum_{i=0}^{k-1} P(A_d(t) = i)} \tag{13}$$

and

$$\lambda_L(X_{d-k+1:d}|X_{s:d}) = \lim_{t\downarrow 0}\frac{P(X_{d-k+1:d}\leq t)}{P(X_{s:d}\leq t)} = \lim_{t\downarrow 0}\frac{\sum_{i=0}^{k-1}P(A_d(t)=i)}{1-\sum_{i=0}^{s-1}P(B_d(t)=i)} \tag{14}$$

we just have to relate $P(B_d(t)=i)$ and $P(A_d(t)=i)$ with function $C$ and with function $\overline{C}$. Observe that,

$$\begin{aligned} P(B_d(t)=i) &= \sum_{I\in F_i} P\Big(\bigcap_{j\in I} F_{X_j}(X_j)\leq t, \bigcap_{j\in \overline{I}} F_{X_j}(X_j) > t\Big) \\ &= \sum_{I\in F_i}\sum_{J\subset \overline{I}}(-1)^{|J|}P\Big(\bigcap_{j\in I} F_{X_j}(X_j)\leq t, \bigcap_{j\in J} F_{X_j}(X_j)\leq t\Big) \\ &= \sum_{I\in F_i}\sum_{J\subset \overline{I}}(-1)^{|J|}C(t^{\mathbb{1}\{1\in I\cup J\}},...,t^{\mathbb{1}\{d\in I\cup J\}}), \end{aligned} \tag{15}$$

as well as,

$$\begin{aligned} P(B_d(t)=i) &= \sum_{I\in F_i}\sum_{J\subset I}(-1)^{|J|}P\Big(\bigcap_{j\in J} F_{X_j}(X_j) > t, \bigcap_{j\in \overline{I}} F_{X_j}(X_j) > t\Big) \\ &= \sum_{I\in F_i}\sum_{J\subset I}(-1)^{|J|}\overline{C}((1-t)^{\mathbb{1}\{1\in \overline{I}\cup J\}},...,(1-t)^{\mathbb{1}\{d\in \overline{I}\cup J\}}). \end{aligned}$$

Analogously we derive $P(A_d(t)=i)$ from $C$ and $\overline{C}$. □

Since $A_d(t) = d - B_d(t)$, we can in practice choose other summations different from those considered in the previous Proposition. This choice can be decided by $\min\{s-1, d-s\}$ and $\min\{k-1, d-k\}$, in the sense of having fewer terms to add.

Next result relates the $s,k$-extremal coefficients with the tail dependence coefficients of sub-vectors of $(X_1,...,X_d)$ and has as a particular case the Proposition 1 in Frahm (2006, [6]) for a random pair $(X_1,X_2)$:

$$\lambda_U(X_{1:2}|X_{2:2}) = \frac{\lambda_U(X_1|X_2)}{2-\lambda_U(X_1|X_2)} \tag{16}$$

$$\lambda_L(X_{2:2}|X_{1:2}) = \frac{\lambda_L(X_1|X_2)}{2-\lambda_L(X_1|X_2)} \tag{17}$$

**Proposition 2.2** *Denote $i(A)$ a fixed element of the set $A$. We have*

$$\lambda_U(X_{s:d}|X_{d-k+1:d}) = \frac{\sum_{0\leq i\leq s-1}\sum_{I\in F_i}\sum_{J\subset I}(-1)^{|J|}\lambda_U\Big(\min_{j\in \overline{I}\cup J}F_{X_j}(X_j)|F_{X_{i(\overline{I}\cup J)}}(X_{i(\overline{I}\cup J)})\Big)}{1-\sum_{0\leq i\leq k-1}\sum_{I\in F_i}\sum_{J\subset \overline{I}}(-1)^{|J|}\lambda_U\Big(\min_{j\in I\cup J}F_{X_j}(X_j)|F_{X_{i(I\cup J)}}(X_{i(I\cup J)})\Big)} \tag{18}$$

*and*

$$\lambda_L(X_{d-k+1:d}|X_{s:d}) = \frac{\sum_{0\leq i\leq k-1}\sum_{I\in F_i}\sum_{J\subset I}(-1)^{|J|}\lambda_L\Big(\max_{j\in \overline{I}\cup J}F_{X_j}(X_j)|F_{X_{i(\overline{I}\cup J)}}(X_{i(\overline{I}\cup J)})\Big)}{1-\sum_{0\leq i\leq s-1}\sum_{I\in F_i}\sum_{J\subset \overline{I}}(-1)^{|J|}\lambda_L\Big(\max_{j\in I\cup J}F_{X_j}(X_j)|F_{X_{i(I\cup J)}}(X_{i(I\cup J)})\Big)}, \tag{19}$$

*provided the existence of the limits corresponding to coefficients $\lambda_U$ and $\lambda_L$ of the terms.*

**Proof.** In order to derive (18), just consider in (10) that

$$\begin{aligned} &\lim_{t\uparrow 1}\overline{C}((1-t)^{\mathbb{1}\{1\in\overline{I}\cup J\}},...,(1-t)^{\mathbb{1}\{d\in\overline{I}\cup J\}}) \\ = &\lim_{t\uparrow 1}P\Big(\min_{j\in\overline{I}\cup J}F_{X_j}(X_j)>t|F_{X_{i(\overline{I}\cup J)}}\big(X_{i(\overline{I}\cup J)}\big)>t\Big)\cdot(1-t) \end{aligned}$$

and perform an analogous reasoning for the terms in denominator.
Now regarding (19), we can consider in (11) that

$$\lim_{t\downarrow 0}C(t^{\mathbb{1}\{1\in\overline{I}\cup J\}},...,t^{\mathbb{1}\{d\in\overline{I}\cup J\}})=\lim_{t\downarrow 0}P\Big(\max_{j\in\overline{I}\cup J}F_{X_j}(X_j)\leq t|F_{X_{i(\overline{I}\cup J)}}\big(X_{i(\overline{I}\cup J)}\big)\leq t\Big)\cdot t$$

and apply the same to the terms of the denominator. □

Observe that $\lambda_U\big(\min_{j\in A}F_{X_j}(X_j)|F_{X_{i(A)}}\big(X_{i(A)}\big)\big)$ and $\lambda_L\big(\max_{j\in A}F_{X_j}(X_j)|F_{X_{i(A)}}\big(X_{i(A)}\big)\big)$ are, respectively, the *upper-orthant tail dependence coefficient*, $\tau_{i(A)}^{C_A}$, and the *lower-orthant tail dependence coefficient*, $\varsigma_{i(A)}^{C_A}$, considered in Li (2009, [16]), where $C_A$ denotes de copula function of the sub-vector of $(X_1,...,X_d)$ with r.v.'s indexed in set $A$. In that work it was proved that, for all $A\subset\{1,...,d\}$,

$$\tau_i^{C_A}=\tau_i^C/\tau_A^C=\varsigma_i^{\overline{C}_A}\text{ and }\varsigma_i^{C_A}=\varsigma_i^C/\varsigma_A^C=\tau_i^{\overline{C}_A},$$

where

$$\tau_A^C=\lambda_U\Big(\min_{i\in\overline{A}}F_{X_i}(X_i)|\min_{i\in A}F_{X_i}(X_i)\Big)\text{ and }\varsigma_A^C=\lambda_L\Big(\max_{i\in\overline{A}}F_{X_i}(X_i)|\max_{i\in A}F_{X_i}(X_i)\Big).$$

Now we apply Proposition 2.1 in the calculation of the $s,k$-extremal coefficients for Marshall-Olkin distributions, using some results in Li (2006, [14]; 2008, [15]) concerning $\tau_A^C$ and $\varsigma_A^C$.

**Example 1** *For each $J\subset\{1,...,d\}$, let $\lambda_J$ be a positive constant and*

$$\nu(A)=\sum_{J:A\subset J}\lambda_J,\ A\subset\{1,...,d\}.$$

*Let $Z_J\frown$ Exponential$(\lambda_J)$ and assume that $\{Z_J\}_{J\subset\{1,...,d\}}$ are independent variables. Consider the d-dimensional random vector $\mathbf{X}=(X_1,...,X_d)$, where*

$$X_i=\min\{Z_J:i\in J\},\ i=1,...,d,$$

*which has Marshall-Olkin distribution (Marshall and Olkin 1967, [17]). It holds (expression (2.3) in Li ( 2006, [16])),*

$$\tau_{i(A)}^{\overline{C}_A}=\min_{i\in A}\frac{\nu(A)}{\nu(i)}=\varsigma_{i(A)}^{C_A}.$$

*and then*

$$\lambda_L(X_{d-k+1:d}|X_{s:d})=\frac{\displaystyle\sum_{0\leq i\leq k-1}\sum_{I\in F_i}\sum_{J\subset I}(-1)^{|J|}\frac{\nu(\overline{I}\cup J)}{\max_{j\in\overline{I}\cup J}\nu(j)}}{\displaystyle-\sum_{\emptyset\neq J\subset\{1,...,d\}}(-1)^{|J|}\frac{\nu(J)}{\max_{j\in J}\nu(j)}-\sum_{1\leq i\leq s-1}\sum_{I\in F_i}\sum_{J\subset\overline{I}}(-1)^{|J|}\frac{\nu(I\cup J)}{\max_{j\in I\cup J}\nu(j)}}.$$

*In order to have endurable calculations without a computer we will take $d=3$ and $\nu(i)=1$, $i=1,2,3$.*

*We obtain*

$$\lambda_L(X_{3-k+1:3}|X_{s:3}) = \begin{cases} \dfrac{\lambda_{\{1,2,3\}}}{3 - \sum\limits_{1\le i<j\le 3} \lambda_{\{i,j\}} + 2\,\lambda_{\{1,2,3\}}} & , s = k = 1 \\ \dfrac{\lambda_{\{1,2,3\}} + \sum\limits_{1\le i<j\le 3} \lambda_{\{i,j\}}}{3 - \sum\limits_{1\le i<j\le 3} \lambda_{\{i,j\}} + 2\,\lambda_{\{1,2,3\}}} & , k = 2, s = 1 \\ \dfrac{\lambda_{\{1,2,3\}}}{\sum\limits_{1\le i<j\le 3} \lambda_{\{i,j\}} + 8\,\lambda_{\{1,2,3\}}} & , k = 1, s = 2 \end{cases}$$

*Following Li (2008, [15]), let*

$$\theta_A = \sum_{i=1}^{d} \sum_{I\in F_i} (-1)^{|I|+1} \frac{\nu(I)}{\max_{j\in I} \nu(j)} - \sum_{\emptyset\neq I\subset A} (-1)^{|I|+1} \frac{\nu(I)}{\max_{j\in I} \nu(j)}.$$

*It holds, for each* $\emptyset \neq J \subset \{1, ..., d\}$,

$$\tau_A^C = \begin{cases} 0 & , \text{ if } \theta_A > 0 \\ 1 & , \text{ if } \theta_A = 0. \end{cases}$$

*Note that if* $B \subset A$ *then* $\tau_B^C \le \tau_A^C$. *Then, if* $\tau_i^C > 0$ *for each* $i \in \{1, ..., d\}$, *we have*

$$\begin{aligned} \lambda_U(X_{s:d}|X_{d-k+1:d}) &= \frac{\sum\limits_{i=0}^{s-1} \sum\limits_{I\in F_i} \sum\limits_{J\subset I} (-1)^{|J|} \tau^C_{i(\overline{I}\cup J)}/\tau^C_{\overline{I}\cup J}}{-\sum\limits_{\emptyset\neq J\subset\{1,\ldots,d\}} (-1)^{|J|} \tau^C_{i(J)}/\tau^C_J - \sum\limits_{i=1}^{k-1} \sum\limits_{I\in F_i} \sum\limits_{J\subset \overline{I}} (-1)^{|J|} \tau^C_{i(I\cup J)}/\tau^C_{I\cup J}} \\ &= \frac{(-1)^0 + \sum\limits_{i=1}^{s-1} \sum\limits_{I\in F_i} \sum\limits_{J\subset I} (-1)^{|J|}}{-\sum\limits_{\emptyset\neq J\subset\{1,\ldots,d\}} (-1)^{|J|} - \sum\limits_{i=1}^{k-1} \sum\limits_{I\in F_i} \sum\limits_{J\subset \overline{I}} (-1)^{|J|}} \\ &= \frac{1+0}{-(-1)+0} = 1\,. \end{aligned}$$

□

Next, we see that the $s, k$-extremal coefficients do not decrease when reducing the distance between $s$ and $d-k+1$. Such intuitive result allows to conclude the total dependence between any order statistics, $X_{s:d}$ and $X_{d-k+1:d}$, from the total dependence between $X_{1:d}$ and $X_{d:d}$.

**Proposition 2.3** *Both coefficients,* $\lambda_U(X_{s:d}|X_{d-k+1:d})$ *and* $\lambda_L(X_{d-k+1:d}|X_{s:d})$, *are non decreasing functions of* $s$ *and* $k$.

**Proof.** Just observe that, if $s < s'$ and $k < k'$, then

$$P(X_{s:d} > t) \le P(X_{s':d} > t) \text{ and } P(X_{d-k+1:d} > t) \ge P(X_{d-k'+1:d} > t). \quad \square$$

**Proposition 2.4** *If* $\mathbf{Y}$ *is a sub-vector of* $\mathbf{X}$ *with dimension* $d-1$, *then the* $s, k$*-extremal coefficients of* $\mathbf{Y}$ *are greater or equal to the corresponding coefficients of* $\mathbf{X}$, *for any* $1 \le s < d-k \le d-1$.

**Proof.** Let $\mathbf{Y}$ be a sub-vector of $\mathbf{X}$ of dimension $d-1$. For $s$ and $k$ such that $1 \leq s < d-1$, $1 < d-1-k+1 \leq d-1$ and $s < d-k$, we have

$$Y_{s:d-1} \geq X_{s:d} \text{ and } Y_{d-k:d-1} \leq X_{d-k+1:d},$$

since when we eliminate one r.v. in $\mathbf{X}$, none of the lower order statistics decreases and none of the upper order statistics increases. Therefore

$$\lambda_U(X_{s:d}|X_{d-k+1:d}) = \lim_{t\uparrow 1} \frac{P(X_{s:d} > t)}{P(X_{d-k+1:d} > t)} \leq \lim_{t\uparrow 1} \frac{P(Y_{s:d-1} > t)}{P(Y_{d-1-k+1:d-1} > t)} = \lambda_U(Y_{s:d-1}|Y_{d-1-k+1:d-1})$$

and

$$\lambda_L(X_{d-k+1:d}|X_{s:d}) = \lim_{t\downarrow 0} \frac{P(X_{d-k+1:d} \leq t)}{P(X_{s:d} \leq t)} \leq \lim_{t\downarrow 0} \frac{P(Y_{d-1-k+1:d-1} \leq t)}{P(Y_{s:d-1} \leq t)} = \lambda_L(Y_{d-1-k+1:d-1}|Y_{s:d-1}). \ \square$$

Now we define the extremal dependence matrix $\Lambda = [\lambda_{ij}]_{i,j=1,\ldots,d}$ with

$$\lambda_{ij} = \begin{cases} 1 = \lambda_U(X_{i:d}|X_{i:d}) = \lambda_L(X_{i:d}|X_{i:d}) & , i = j \\ \lambda_L(X_{i:d}|X_{j:d}) & , i > j \\ \lambda_U(X_{i:d}|X_{j:d}) & , i < j \end{cases}$$

and analyze conditions for its symmetry. Vectors $\mathbf{X}$ with transpositional symmetric, comonotonic or independent copula lead to symmetric $\Lambda$.

**Proposition 2.5** *If $\mathbf{X}$ has symmetric copula, in the sense that $C(u_1, ..., u_d) = \overline{C}(u_1, ..., u_d)$, $\forall (u_1, ..., u_d) \in [0,1]^d$, then $\lambda_U(X_{s:d}|X_{d-s+1:d}) = \lambda_L(X_{d-s+1:d}|X_{s:d})$ for all $1 \leq s \leq [(d+1)/2]$.*

**Proof.** Given (13) and (14), it suffices to prove that $\lim_{t\uparrow 1} P(B_d(t) = i) = \lim_{t\downarrow 0} P(A_d(t) = i)$. Applying (15) and then the symmetry assumption, we have that

$$\lim_{t\uparrow 1} P(B_d(t) = i) = \lim_{t\uparrow 1} \sum_{I\in F_i} \sum_{J\subset \overline{I}} (-1)^{|J|} P\Big(\bigcap_{j\in I} F_{X_j}(X_j) \leq t, \bigcap_{j\in J} F_{X_j}(X_j) \leq t\Big)$$

$$= \lim_{t\downarrow 0} \sum_{I\in F_i} \sum_{J\subset \overline{I}} (-1)^{|J|} P\Big(\bigcap_{j\in I\cup J} F_{X_j}(X_j) \leq 1-t\Big) = \lim_{t\downarrow 0} \sum_{I\in F_i} \sum_{J\subset \overline{I}} (-1)^{|J|} P\Big(\bigcap_{j\in I\cup J} F_{X_j}(X_j) > t\Big)$$

$$= \lim_{t\downarrow 0} P(A_d(t) = i) \, . \ \square$$

**Proposition 2.6** *If $\mathbf{X}$ has copula function $C(u_1, ..., u_d) = \min\{u_1, ..., u_d\}$ then, for any $1 \leq s < d-k+1 \leq d$, we have*

$$\lambda_U(X_{s:d}|X_{d-k+1:d}) = \lambda_L(X_{d-k+1:d}|X_{s:d}) = 1.$$

**Proof.** In Frahm (2006, [6]) it is proved for $s = k = 1$ (Proposition 6). For the remaining order statistics it follows from inequalities of Proposition 2.3. $\square$

**Proposition 2.7** *If $\mathbf{X}$ has copula function $C(u_1, ..., u_d) = \prod_{i=1}^d u_i$ then, for any $1 \leq s < d-k+1 \leq d$, we have*

$$\lambda_U(X_{s:d}|X_{d-k+1:d}) = \lambda_L(X_{d-k+1:d}|X_{s:d}) = 0.$$

**Proof.** Based on representation (9), we have

$$\begin{aligned} \lambda_U(X_{s:d}|X_{d-k+1:d}) = & \lim_{t\uparrow 1} \frac{\sum_{i=0}^{s-1} \sum_{I\in F_i} \sum_{J\subset \overline{I}} (-1)^{|J|} \, t^{|I\cup J|}}{1 - \sum_{i=0}^{k-1} \sum_{I\in F_i} \sum_{J\subset I} (-1)^{|J|} \, t^{|\overline{I}\cup J|}} \\ = & \lim_{t\uparrow 1} \frac{1 + \sum_{\emptyset\neq J\subset\{1,\ldots,d\}} (-1)^{|J|} \, t^{|J|} + \sum_{i=1}^{s-1} \sum_{I\in F_i} \sum_{J\subset \overline{I}} (-1)^{|J|} \, t^{|I\cup J|}}{1 - t^d - \sum_{i=1}^{k-1} \sum_{I\in F_i} \sum_{J\subset I} (-1)^{|J|} \, t^{|\overline{I}\cup J|}} \, . \end{aligned}$$

Since, $\sum_{\emptyset\neq J\subset A}(-1)^{|J|}|J| = \sum_{i=1}^{|A|}\binom{|A|}{i}(-1)^i\, i = 0$ and $\sum_{J\subset A\neq\emptyset}(-1)^{|J|} = 0$, after applying l'Hospital's rule, we obtain

$$\lambda_U(X_{s:d}|X_{d-k+1:d}) = \frac{\sum_{\emptyset\neq J\subset\{1,\ldots,d\}}(-1)^{|J|}|J| + \sum_{i=1}^{s-1}\sum_{I\in F_i}\sum_{J\subset\overline{I}}(-1)^{|J|}(|I|+|J|)}{-d-\sum_{i=1}^{k-1}\sum_{I\in F_i}\sum_{J\subset I}(-1)^{|J|}(|\overline{I}|+|J|)} = 0\,.$$

It is analogous for coefficient $\lambda_L(X_{d-k+1:d}|X_{s:d})$ if we take representation (12) (just replace $t$ by $1-t$ and exchange $s$ with $k$ in the expressions above). The result for $s=k=1$ is also stated in Frahm (2006, [6]) (Proposition 7). □

We have shown that, for the product copula, r.v.'s $X_{s:d}$ and $X_{d-k+1:d}$ are upper and lower asymptotic tail independent (though the converse may not be true as can be seen in Example 4).

In order to graduate the "strength" of dependence within the case of asymptotic tail independence, Ledford and Tawn [11, 12] (1996, 1997) have introduced a coefficient, $\eta$, usually termed *coefficient of tail dependence*, based on a regularly varying formulation for the bivariate survival copula that states its rate of convergence towards zero.

Here we consider the coefficient of tail dependence also for the lower tail and extended to $d$-dimensional random vectors $(X_1,...,X_d)$. More precisely, we have for the upper tail

$$P(F_{X_1}(X_1)>1-t,...,F_{X_d}(X_d)>1-t)\sim t^{1/\eta_U(\mathbf{X})}L_{\mathbf{X}}^{(U)}(t),\ \text{ as } t\downarrow 0, \tag{20}$$

and for the lower tail,

$$P(F_{X_1}(X_1)<t,...,F_{X_d}(X_d)<t)\sim t^{1/\eta_L(\mathbf{X})}L_{\mathbf{X}}^{(L)}(t),\ \text{ as } t\downarrow 0, \tag{21}$$

where the coefficients of tail dependence, $\eta_U(\mathbf{X})$ and $\eta_L(\mathbf{X})$, take values in the interval $(0,1]$ and $L_{\mathbf{X}}^{(U)}(t)$ and $L_{\mathbf{X}}^{(L)}(t)$ are slowly varying functions at 0, i.e., $L_{\mathbf{X}}^{(i)}(tx)/L_{\mathbf{X}}^{(i)}(t)\to 1$ $(i=L,U)$ for any $x>0$, as $t\downarrow 0$.

Observe that for symmetric copulas (in the sense of Proposition 2.5), relations (20) and (21) are equivalent, where the coefficients and the slowly varying functions coincide, respectively. We give simple examples of vectors satisfying (20) and (21).

**Example 2** *Let $\{Y_i\}_{i=1,\ldots,d+s}$ be a family of i.i.d. r.v.'s with marginal d.f. $F_Y$.*

*(a) Define $\mathbf{X}=(X_1,...,X_d)$ such that, $X_i=\min(Y_i,...,Y_{i+s})$, $i=1,...,d$. Observe that $F_X(x)=P(X_i\leq x)=1-(1-F_Y(x))^{s+1}$, and hence, $F_X^{-1}(x)=F_Y^{-1}(1-(1-x)^{1/(s+1)})$. We have*

$$\begin{aligned} P\big(\cap_{i=1}^d F_X(X_i)>1-t\big) &= P\big(\cap_{i=1}^{s+d} Y_i > F_X^{-1}(1-t)\big) = \big(1-F_Y(F_X^{-1}(1-t))\big)^{s+d} \\ &= \big(1-F_Y(F_Y^{-1}(1-t^{1/(s+1)}))\big)^{s+d} = t^{\frac{s+d}{s+1}} \end{aligned}$$

*and (20) holds with $\eta_U(\mathbf{X})=\frac{s+1}{s+d}$ and $L_{\mathbf{X}}^{(U)}(t)=1$.*

*(b) Now, if we consider $\mathbf{X}$ such that $X_i=\max(Y_i,...,Y_{i+s})$, $i=1,...,d$, we have $F_X(x)=P(X_i\leq x)=F_Y(x)^{s+1}$ and $F_X^{-1}(x)=F_Y^{-1}(x^{1/(s+1)})$. Hence*

$$\begin{aligned} P\big(\cap_{i=1}^d F_{X_i}(X_i)<t\big) &= P\big(\cap_{i=1}^{s+d} Y_i < F_X^{-1}(t)\big) = F_Y\big(F_X^{-1}(t)\big)^{s+d} \\ &= F_Y\big(F_Y^{-1}(t^{1/(s+1)})\big)^{s+d} = t^{\frac{s+d}{s+1}} \end{aligned}$$

*and (21) holds with $\eta_L(\mathbf{X})=\frac{s+1}{s+d}$ and $L_{\mathbf{X}}^{(L)}(t)=1$.* □

Next result relates the $s,k$-extremal coefficients with $\eta_U(\mathbf{X}_A)$ and $\eta_L(\mathbf{X}_A)$, for subsets $A\subset\{1,...,d\}$.

**Proposition 2.8** *Consider notation* $\eta_1 = \max\{\eta_{\mathbf{X}_I} : |I| = d-s+1\}$, $\eta_2 = \max\{\eta_{\mathbf{X}_I} : |I| = k\}$, $\eta_3 = \max\{\eta_{\mathbf{X}_I} : |I| = d-k+1\}$ *and* $\eta_4 = \max\{\eta_{\mathbf{X}_I} : |I| = s\}$. *Under the assumption in (20), we have*

$$\lambda_U(X_{s:d}|X_{d-k+1:d}) \sim t^{1/\eta_1 - 1/\eta_2} L_*(t)\,, \quad \text{as } t \downarrow 0, \tag{22}$$

*where slowly varying function* $L_*(t)$ *is the ratio of the slowly varying functions* $L_1(t)$ *and* $L_2(t)$*, associated to* $\eta_1$ *and* $\eta_2$*, respectively. Under the assumption in (21), we have*

$$\lambda_L(X_{d-k+1:d}|X_{s:d}) \sim t^{1/\eta_3 - 1/\eta_4} L_{**}(t)\,, \quad \text{as } t \downarrow 0, \tag{23}$$

*where slowly varying function* $L_{**}(t)$ *is the ratio of the slowly varying functions* $L_3(t)$ *and* $L_4(t)$*, associated to* $\eta_3$ *and* $\eta_4$*, respectively.*

**Proof.** In order to derive (22), we are going to apply expression (10) of Proposition 2.1. Observe that, for any $A \subset \{1, ..., d\}$,

$$\lim_{t\uparrow 1} \overline{C}((1-t)^{\mathbb{1}\{1\in A\}}, ..., (1-t)^{\mathbb{1}\{d\in A\}}) = \lim_{t\downarrow 0} \overline{C}(t^{\mathbb{1}\{1\in A\}}, ..., t^{\mathbb{1}\{d\in A\}})$$

and

$$\overline{C}(t^{\mathbb{1}\{1\in A\}}, ..., t^{\mathbb{1}\{d\in A\}}) \underset{t\downarrow 0}{\sim} t^{1/\eta_U(\mathbf{X}_A)} L^{(U)}_{\mathbf{X}_A}(t).$$

Hence, as $t \downarrow 0$, we have

$$\lambda_U(X_{s:d}|X_{d-k+1:d}) \underset{t\downarrow 0}{\sim} \frac{\sum\limits_{I:|I|=d-s+1} \overline{C}((1-t)^{\mathbb{1}\{1\in I\}}, ..., (1-t)^{\mathbb{1}\{d\in I\}})}{\sum\limits_{I:|I|=k} \overline{C}((1-t)^{\mathbb{1}\{1\in I\}}, ..., (1-t)^{\mathbb{1}\{d\in I\}})} \underset{t\downarrow 0}{\sim} \frac{\sum\limits_{I:|I|=d-s+1} t^{1/\eta_U(\mathbf{X}_I)} L^{(U)}_{\mathbf{X}_I}(t)}{\sum\limits_{I:|I|=k} t^{1/\eta_U(\mathbf{x}_I)} L^{(U)}_{\mathbf{X}_I}(t)}$$

and the result is straightforward.

Concerning (23), observe that

$$C(t^{\mathbb{1}\{1\in A\}}, ..., t^{\mathbb{1}\{d\in A\}}) \sim t^{1/\eta_L(\mathbf{X}_A)} L^{(L)}_{\mathbf{X}_A}(t)\,, \text{ as } t \downarrow 0,$$

and, considering expression (11) in Proposition 2.1, we have now

$$\lambda_L(X_{d-k+1:d}|X_{s:d}) \underset{t\downarrow 0}{\sim} \frac{\sum\limits_{I:|I|=d-k+1} C(t^{\mathbb{1}\{1\in I\}}, ..., t^{\mathbb{1}\{d\in I\}})}{\sum\limits_{I:|I|=s} C(t^{\mathbb{1}\{1\in I\}}, ..., t^{\mathbb{1}\{d\in I\}})} \underset{t\downarrow 0}{\sim} \frac{\sum\limits_{I:|I|=d-k+1} t^{1/\eta_L(\mathbf{X}_I)} L^{(L)}_{\mathbf{X}_I}(t)}{\sum\limits_{I:|I|=s} t^{1/\eta_L(\mathbf{X}_I)} L^{(L)}_{\mathbf{X}_I}(t)},$$

leading to the result. □

Retaking Example 2 above, for $d = 3$ and $s = 1$, we have, as $t \downarrow 0$,

$$\lambda_U(X_{s:3}|X_{3-k+1:3}) \sim \begin{cases} t & ,\ s = k = 1 \\ t^{1/2} & ,\ s = 1, k = 2 \text{ or } s = 2, k = 1\,. \end{cases}$$

The same asymptotic equivalence holds for $\lambda_L(X_{3-k+1:3}|X_{s:3})$.

In Klüppelberg *et al.* (2008, [10]) it is considered the multivariate *tail dependence function* (see Schmidt and Stadtmuller (2006, [24]) for the bivariate setting), which can be established both for upper and lower tails. The *tail dependence function* is then defined by

$$\lambda^{\mathbf{X}}_U(x_1, ..., x_d) = \lim_{t\downarrow 0} \frac{P(F_{X_1}(X_1) > 1 - tx_1, ..., F_{X_d}(X_d) > 1 - tx_d)}{t}$$

for all $(x_1, ..., x_d)$ with nonnegative components. Hence, for any $\emptyset \neq A \subset \{1, ..., d\}$, we have

$$\lambda_U^{\mathbf{X}_A}(\mathbf{1}_A) = \lim_{t \downarrow 0} \frac{P(\cap_{i \in A} F_{X_i}(X_i) > 1 - t)}{t} = \lim_{t \uparrow 1} \frac{P(\cap_{i \in A} F_{X_i}(X_i) > t)}{1 - t},$$

where $\mathbf{1}_A$ denotes the unit vector with dimension $|A|$.
If we also consider

$$\lambda_L^{\mathbf{X}}(x_1, ..., x_d) = \lim_{t \downarrow 0} \frac{P(F_{X_1}(X_1) \leq t x_1, ..., F_{X_d}(X_d) \leq t x_d)}{t}$$

we have for $\emptyset \neq A \subset \{1, ..., d\}$,

$$\lambda_L^{\mathbf{X}_A}(\mathbf{1}_A) = \lim_{t \downarrow 0} \frac{P(\cap_{i \in A} F_{X_i}(X_i) \leq t)}{t}.$$

Coefficients $\lambda_U^{\mathbf{X}_A}(\mathbf{1}_A)$ and $\lambda_L^{\mathbf{X}_A}(\mathbf{1}_A)$ measure the extremal dependence of $(F_{X_i}(X_i), i \in A)$ around the boundary points, respectively, $\mathbf{1}_A$ and $\mathbf{0}_A$, along the direction of vector $\mathbf{1}_A$.

The $s, k$-extremal coefficients can also incorporate the information contained in these coefficients, as we shall see in the next result.

**Proposition 2.9** *The $s, k$-extremal coefficients satisfy the following relations:*

$$\lambda_U(X_{s:d}|X_{d-k+1:d}) = \frac{\sum_{i=0}^{s-1} \sum_{I \in F_i} \sum_{J \subset I} (-1)^{|J|} \lambda_U^{\mathbf{X}_{\overline{I} \cup J}}(\mathbf{1}_{\overline{I} \cup J})}{-\sum_{\emptyset \neq J \subset \{1,...,d\}} (-1)^{|J|} \lambda_U^{\mathbf{X}_J}(\mathbf{1}_J) - \sum_{i=1}^{k-1} \sum_{I \in F_i} \sum_{J \subset \overline{I}} (-1)^{|J|} \lambda_U^{\mathbf{X}_{I \cup J}}(\mathbf{1}_{I \cup J})} \quad (24)$$

$$\lambda_L(X_{d-k+1:d}|X_{s:d}) = \frac{\sum_{i=0}^{k-1} \sum_{I \in F_i} \sum_{J \subset I} (-1)^{|J|} \lambda_L^{\mathbf{X}_{\overline{I} \cup J}}(\mathbf{1}_{\overline{I} \cup J})}{-\sum_{\emptyset \neq J \subset \{1,...,d\}} (-1)^{|J|} \lambda_L^{\mathbf{X}_J}(\mathbf{1}_J) - \sum_{i=1}^{s-1} \sum_{I \in F_i} \sum_{J \subset \overline{I}} (-1)^{|J|} \lambda_L^{\mathbf{X}_{I \cup J}}(\mathbf{1}_{I \cup J})}, \quad (25)$$

*provided the ratios are defined.*

**Proof.** In order to derive (24) we use representation (10) in Proposition 2.1 and divide both numerator and denominator by $1 - t$. With respect to (25), we use representation (11) in Proposition 2.1 and divide both numerator and denominator by $t$. □

About the above ratios we remark that, as limits of nondecreasing functions, when $\lambda_U^{\mathbf{X}_A}(\mathbf{x}_A)$ and $\lambda_L^{\mathbf{X}_A}(\mathbf{x}_A)$ exist they are nondecreasing and then, for $A \supset B$, $\lambda_i^{\mathbf{X}_A}(\mathbf{x}_A) \leq \lambda_i^{\mathbf{X}_B}(\mathbf{x}_B)$, $i = U, L$. Moreover, $\lambda_U^{\mathbf{X}_A}(\mathbf{x}_A)$ and $\lambda_L^{\mathbf{X}_A}(\mathbf{x}_A)$ are nonzero everywhere if they do not vanish in a single point. Therefore, if $\lambda_U^{\mathbf{X}}(\mathbf{x}) > 0$ ($\lambda_L^{\mathbf{X}}(\mathbf{x}) > 0$ ) for some $\mathbf{x}$ then all the terms in (24) ((25)) are non null. Otherwise, if $X_i$ and $X_j$ are upper tail-independent, for each $1 \leq i < j \leq d$, that is, $\lambda_U(X_i|X_j) = \lambda_U^{(X_i, X_j)}(1, 1) = 0$, then the ratio in (24) ( *mutatis mutandis* for (25) ) is not defined.

Klüppelberg *et al.* (2008, [10]) give the explicit formula of function $\lambda_U^X(x_1, ..., x_d)$ for random vectors $\mathbf{X}$ with elliptical distribution having "generating variate" regularly varying with index $\alpha > 0$ and correlation matrix $R = \Lambda \Lambda^T$, where $\Lambda$ is a deterministic $d \times d$ matrix with full rank. Denoting $\Lambda_i$ the $i$-th row of $\Lambda$ and $F_U$ the uniform distribution on the unit sphere $\mathcal{S}_d = \{(u_1, ..., u_d) = \mathbf{u} \in \mathbb{R}^d : \sum_{i=1}^d u_i^2 = 1\}$, it follows from its Theorem 5.1, that

$$\lambda_U^{\mathbf{X}_A}(\mathbf{1}_A) = \frac{\int_{\{\mathbf{u} \in \mathcal{S}_d; \Lambda_i \mathbf{u} > 0, i=1,...,d\}} \min_{i \in A} (\Lambda_i \mathbf{u})^\alpha dF_U(\mathbf{u})}{\int_{\{\mathbf{u} \in \mathcal{S}_d; \Lambda_1 \mathbf{u} > 0\}} (\Lambda_1 \mathbf{u})^\alpha dF_U(\mathbf{u})},$$

for each $A \subset \{1, ..., d\}$. Denoting $\Xi = \{\mathbf{u} \in \mathcal{S}_d; \Lambda_j \mathbf{u} > 0, j = 1, ..., d\}$, we then obtain

$$\lambda_U(X_{s:d}|X_{d-k+1:d}) = \frac{\sum_{i=0}^{s-1} \sum_{I \in F_i} \sum_{J \subset I} (-1)^{|J|} \int_\Xi \min_{j \in \overline{I} \cup J} (\Lambda_j \mathbf{u})^\alpha dF_U(\mathbf{u})}{-\sum_{\emptyset \neq J \subset \{1,...,d\}} (-1)^{|J|} \int_\Xi \min_{j \in J} (\Lambda_j \mathbf{u})^\alpha dF_U(\mathbf{u}) - \sum_{i=1}^{k-1} \sum_{I \in F_i} \sum_{J \subset \overline{I}} (-1)^{|J|} \int_\Xi \min_{j \in I \cup J} (\Lambda_j \mathbf{u})^\alpha dF_U(\mathbf{u})}.$$

For different approaches in computing $\lambda_L(X_{d:d}|X_{1:d}) = \lambda_U(X_{1:d}|X_{d:d})$ for elliptical distributions see Frahm (2006, [6]).

# 3 Extremal coefficients for multivariate extreme value distributions

Consider $\mathbf{X} = (X_1, ..., X_d)$ with multivariate extreme value distribution (MEV). Then there exists a constant $\epsilon(\mathbf{X}) \in [1, d]$ such that, for all $(u_1, ..., u_d) \in [0, 1]^d$, we have $C(u, ..., u) = u^{\epsilon(\mathbf{X})}$ (Tiago de Oliveira 1962/63, [28]; Smith 1990, [26]). If $\mathbf{X}_J$ is a sub-vector of $\mathbf{X}$ with r.v.'s indexed in $J$, then $\mathbf{X}_J$ has also MEV distribution and we denote the respective extremal coefficient $\epsilon$ by $\epsilon(\mathbf{X}_J)$, where $\epsilon(\mathbf{X}_J) \in [1, |J|]$.

As the coefficient of tail dependence $\eta$, the extremal coefficient $\epsilon$ is an extension of the independent components case for MEV distributions.

It is already known that if $\mathbf{X} = (X_1, X_2)$ has bivariate extreme value distribution then $\lambda_U(X_1|X_2) = 2 - \epsilon(\mathbf{X})$ and, by (16), $\lambda_U(X_{1:2}|X_{2:2}) = \frac{2-\epsilon(\mathbf{X})}{\epsilon(\mathbf{X})}$.

The next result suggests an extension of these relations, and it is calculated the upper $s, k$-extremal coefficient from coefficients $\epsilon$ of sub-vectors of $\mathbf{X}$.

**Proposition 3.1** *If $\mathbf{X}$ has MEV distribution then, for any $1 \leq s < d - k + 1 \leq d$,*

$$\lambda_U(X_{s:d}|X_{d-k+1:d}) = \frac{\sum_{\emptyset \neq J \subset \{1,\ldots,d\}} (-1)^{|J|} \epsilon(\mathbf{X}_J) + \sum_{i=1}^{s-1} \sum_{I \in F_i} \sum_{J \subset \overline{I}} (-1)^{|J|} \epsilon(\mathbf{X}_{I \cup J})}{-\sum_{i=0}^{k-1} \sum_{I \in F_i} \sum_{J \subset I} (-1)^{|J|} \epsilon(\mathbf{X}_{\overline{I} \cup J})}, \tag{26}$$

*provided the ratio is defined.*

**Proof.** Observe that

$$\lambda_U(X_{s:d}|X_{d-k+1:d}) = \lim_{t \uparrow 1} \frac{1 + \sum_{\emptyset \neq J \subset \{1,\ldots,d\}} (-1)^{|J|} t^{\epsilon(\mathbf{X}_J)} + \sum_{i=1}^{s-1} \sum_{I \in F_i} \sum_{J \subset \overline{I}} (-1)^{|J|} t^{\epsilon(\mathbf{X}_{I \cup J})}}{1 - \sum_{i=0}^{k-1} \sum_{I \in F_i} \sum_{J \subset I} (-1)^{|J|} t^{\epsilon(\mathbf{X}_{\overline{I} \cup J})}}$$

and the l'Hospital's rule leads to the result. □

If $X$ has totally dependent marginals, then expression (26) comes

$$\lambda_U(X_{s:d}|X_{d-k+1:d}) = \frac{\sum_{\emptyset \neq J \subset \{1,\ldots,d\}} (-1)^{|J|} + \sum_{i=1}^{s-1} \sum_{I \in F_i} \sum_{J \subset \overline{I}} (-1)^{|J|}}{-(-1)^0 - \sum_{i=1}^{k-1} \sum_{I \in F_i} \sum_{J \subset I} (-1)^{|J|}} = \frac{-1 + \sum_{i=1}^{s-1} \sum_{I \in F_i} 0}{-1 - \sum_{i=1}^{k-1} \sum_{I \in F_i} 0} = 1$$

which is the result obtained in Proposition 2.6, and if $X$ has independent marginals, then

$$\lambda_U(X_{s:d}|X_{d-k+1:d}) = \frac{\sum_{\emptyset \neq J \subset \{1,\ldots,d\}} (-1)^{|J|} |J| + \sum_{i=1}^{s-1} \sum_{I \in F_i} \sum_{J \subset \overline{I}} (-1)^{|J|} |I \cup J|}{-\sum_{i=0}^{k-1} \sum_{I \in F_i} \sum_{J \subset I} (-1)^{|J|} |\overline{I} \cup J|},$$

which is null as we have seen in Proposition 2.7.

If $A \subset B \subset \{1, ..., d\}$ then $\epsilon(\mathbf{X}_A) \leq \epsilon(\mathbf{X}_B)$. In Schlatter and Tawn (2002, [21]) it is presented other consistent conditions for coefficients $\epsilon(\mathbf{X}_A)$, $A \subset \{1, ..., d\}$.
Observe that, if we take $s = k = 1$ in Proposition 3.1, we obtain

$$\lambda_U(X_{1:d}|X_{d:d}) = \frac{\sum_{\emptyset \neq J \subset \{1,\ldots,d\}} (-1)^{|J|} \epsilon(\mathbf{X}_J)}{-\epsilon(\mathbf{X})}$$

having as a particular case the already mentioned bivariate situation.

**Example 3** *Consider $\mathbf{X}=(X_1,X_2,X_3)$ with Gumbel distribution*

$$F(x_1,x_2,x_3)=\exp\big(-(e^{-\alpha x_1}+e^{-\alpha x_2}+e^{-\alpha x_3})^{1/\alpha}\big),$$

*where $\alpha\geq 1$. If $\alpha=1$ then $\mathbf{X}$ has independent marginals and hence the upper $s,k$-extremal coefficients are null. In case $\alpha>1$, for $|A|=1$ we have $\epsilon(\mathbf{X}_A)=1$, for $|A|=2$ we have $\epsilon(\mathbf{X}_A)=\frac{-\ln F_{\mathbf{X}_A}(x,x)}{-\ln F_{X_1}(x)}=2^{1/\alpha}$ and $\epsilon(\mathbf{X})=3^{1/\alpha}$. Therefore, applying (26) we obtain,*

$$\lambda_U(X_{s:3}|X_{3-k+1:3})=\begin{cases}\frac{3(2^{1/\alpha}-1)-3^{1/\alpha}}{-3^{1/\alpha}} & ,\ s=k=1\\ \frac{3(2^{1/\alpha}-1)-3^{1/\alpha}}{2\times 3^{1/\alpha}-3\times 2^{1/\alpha}} & ,\ s=1,k=2\\ \frac{-3\times 2^{1/\alpha}+2\times 3^{1/\alpha}}{-3^{1/\alpha}} & ,\ s=2,k=1\end{cases}$$

*Note that, if $\alpha=1$, the first and third ratios allow us to recover the value $0$ while the second one is not defined, although we know that the respective $1,2$-extremal coefficient is also null.* □

Next we will see how the $s,k$-extremal coefficients can incorporate the information of dependence coming from the spectral measure $S$ of multivariate extreme value distribution.

If $\mathbf{X}$ has MEV distribution with marginals unit Fréchet, then (Resnick 1987, [20]) there exists a finite measure $S$ on the unit sphere $\mathcal{S}_d$, satisfying $\int_{\mathcal{S}_d}u_i\,dS(\mathbf{u})=1$, $i=1,...,d$, and such that

$$F_{\mathbf{X}_A}(\mathbf{x}_A)=\exp\left(-\int_{\mathcal{S}_d}\max_{i\in A}\frac{u_i}{x_i}\,dS(\mathbf{u})\right), \tag{27}$$

for all $(x_1,...,x_d)\in\mathbb{R}_+^d$ and $A\subset\{1,...,d\}$, where $\mathbf{x}_A$ denotes the sub-vector of $\mathbf{x}=(x_1,...,x_d)$ with indices in $A$.

**Proposition 3.2** *If $\mathbf{X}$ has MEV distribution with unit Fréchet marginals and spectral measure $S$, then*

$$\lambda_U(X_{s:d}|X_{d-k+1:d})=\frac{\sum_{\emptyset\neq J\subset\{1,...,d\}}(-1)^{|J|}\int_{\mathcal{S}_d}\max_{j\in J}u_j\,dS(\mathbf{u})+\sum_{i=1}^{s-1}\sum_{I\in F_i}\sum_{J\subset\overline{I}}(-1)^{|J|}\int_{\mathcal{S}_d}\max_{j\in I\cup J}u_j\,dS(\mathbf{u})}{-\sum_{i=0}^{k-1}\sum_{I\in F_i}\sum_{J\subset I}(-1)^{|J|}\int_{\mathcal{S}_d}\max_{j\in\overline{I}\cup J}u_j\,dS(\mathbf{u})} \tag{28}$$

*and*

$$\lambda_L(X_{d-k+1:d}|X_{s:d})=\frac{\sum_{i=0}^{k-1}\sum_{I\in F_i}\sum_{J\subset I}(-1)^{|J|}\delta_S(\overline{I}\cup J)}{-\sum_{\emptyset\neq J\subset\{1,...,d\}}(-1)^{|J|}\delta_S(J)-\sum_{i=1}^{s-1}\sum_{I\in F_i}\sum_{J\subset\overline{I}}(-1)^{|J|}\delta_S(I\cup J)}, \tag{29}$$

*provided the ratios are defined, where*

$$\delta_S(A)=\begin{cases}1 & ,\ \text{if}\ \int_{\mathcal{S}_d}\big(\max_{i\in A}u_i-\min_{i=1,...,d}u_i\big)\,dS(\mathbf{u})=0\\ 0 & ,\ \text{if}\ \int_{\mathcal{S}_d}\big(\max_{i\in A}u_i-\min_{i=1,...,d}u_i\big)\,dS(\mathbf{u})>0.\end{cases}$$

**Proof.** Consider notation $\mathcal{I}_1(A) = \int_{\mathcal{S}_d} \max_{j\in A} u_j \, dS(\mathbf{u})$. From expression (9) in Proposition 2.1 and applying (27), we have

$$\lambda_U(X_{s:d}|X_{d-k+1:d}) =$$
$$= \lim_{x\to\infty} \frac{1 + \sum_{\emptyset\neq J\subset\{1,\dots,d\}} (-1)^{|J|} \exp\left(-\frac{1}{x}\mathcal{I}_1(J)\right) + \sum_{i=1}^{s-1}\sum_{I\in F_i}\sum_{J\subset \overline{I}} (-1)^{|J|} \exp\left(-\frac{1}{x}\mathcal{I}_1(I\cup J)\right)}{1 - \sum_{i=0}^{k-1}\sum_{I\in F_i}\sum_{J\subset I} (-1)^{|J|} \exp\left(-\frac{1}{x}\mathcal{I}_1(\overline{I}\cup J)\right)}$$
$$= \lim_{x\to\infty} \frac{\sum_{\emptyset\neq J\subset\{1,\dots,d\}} (-1)^{|J|} \frac{1}{x^2}\mathcal{I}_1(J) \exp\left(-\frac{1}{x}\mathcal{I}_1(J)\right) + \sum_{i=1}^{s-1}\sum_{I\in F_i}\sum_{J\subset \overline{I}} (-1)^{|J|} \frac{1}{x^2}\mathcal{I}_1(I\cup J) \exp\left(-\frac{1}{x}\mathcal{I}_1(I\cup J)\right)}{\sum_{i=0}^{k-1}\sum_{I\in F_i}\sum_{J\subset I} (-1)^{|J|} \frac{1}{x^2}\mathcal{I}_1(\overline{I}\cup J) \exp\left(-\frac{1}{x}\mathcal{I}_1(\overline{I}\cup J)\right)}$$

and the result is straightforward since the exponential functions converge to 1.
Regarding the second statement, consider notation $\mathcal{I}_2(A) = \int_{\mathcal{S}_d} \left(\max_{i\in A} u_i - \min_{i=1,\dots,d} u_i\right) dS(\mathbf{u})$. If we apply (27) in formulation (11) of Proposition 2.1 and then divide both numerator and denominator by $\int_{\mathcal{S}_d} \min_{i=1,\dots,d} u_i \, dS(\mathbf{u})$, we obtain

$$\lambda_L(X_{d-k+1:d}|X_{s:d}) = \lim_{x\to 0} \frac{\sum_{i=0}^{k-1}\sum_{I\in F_i}\sum_{J\subset I} (-1)^{|J|} \exp\left(-\frac{1}{x}\mathcal{I}_2(\overline{I}\cup J)\right)}{-\sum_{\emptyset\neq J\subset\{1,\dots,d\}} (-1)^{|J|} \exp\left(-\frac{1}{x}\mathcal{I}_2(J)\right) - \sum_{i=1}^{s-1}\sum_{I\in F_i}\sum_{J\subset \overline{I}} (-1)^{|J|} \exp\left(-\frac{1}{x}\mathcal{I}_2(I\cup J)\right)},$$

where each of the exponential functions converge to 0 or is equal to 1. □

Note that if all the values $\delta_S(.)$ in the previous ratio are equal to 1, we have $\lambda_L(X_{d-k+1:d}|X_{s:d})$=1:

$$\frac{\sum_{i=0}^{k-1}\sum_{I\in F_i}\sum_{J\subset I} (-1)^{|J|}}{-\sum_{\emptyset\neq J\subset\{1,\dots,d\}} (-1)^{|J|} - \sum_{i=1}^{s-1}\sum_{I\in F_i}\sum_{J\subset \overline{I}} (-1)^{|J|}} = \frac{(-1)^0 + \sum_{i=1}^{k-1}\sum_{I\in F_i}\sum_{J\subset I} (-1)^{|J|}}{-(-1) - \sum_{i=1}^{s-1}\sum_{I\in F_i}\sum_{J\subset \overline{I}} (-1)^{|J|}} = \frac{1 + \sum_{i=1}^{k-1}\sum_{I\in F_i} 0}{1 - \sum_{i=1}^{s-1}\sum_{I\in F_i} 0} = 1$$

By applying (27), the expression (28) can also be derived from (26) in Proposition 3.1 when its ratio is defined.

Suppose that $\mathbf{Y}$ has d.f. in the max-domain of attraction of the MEV distribution of $\mathbf{X}$ and that, without loss of generality, $\mathbf{Y}$ has marginal d.f. unit Pareto, $F_{Y_i}(x) = 1 - x^{-1}$, $x \geq 1$, $i = 1, \dots, d$, and $\mathbf{X}$ has marginal d.f. unit Fréchet, $F_{X_i}(x) = \exp(-x^{-1})$, $x > 0$, $i = 1, \dots, d$. We show in the following result that the respective $s, k$-extremal coefficients coincide in both vectors.

**Proposition 3.3** *If* $F^n_{Y_1}(nx) = (1 - (nx)^{-1})^n \underset{n\to\infty}{\longrightarrow} \exp(-x^{-1}) = F_{X_1}(x)$ *and* $F^n_{\mathbf{Y}}(nx_1, \dots, nx_d) \underset{n\to\infty}{\longrightarrow} F_{\mathbf{X}}(x_1, \dots, x_d)$ *at continuity points of the MEV d.f.* $F_{\mathbf{X}}$*, then for any* $1 \leq s < d - k + 1 \leq d$*, we have*

$$\lambda_U(X_{s:d}|X_{d-k+1:d}) = \lambda_U(Y_{s:d}|Y_{d-k+1:d}).$$

**Proof.** From representation (9), it suffices to show that, for any $\emptyset \neq A \subset \{1, \dots, d\}$, we have

$$\lim_{x\to\infty} C^X\left(F_{X_1}(x)^{\mathbb{1}\{1\in A\}}, \dots, F_{X_d}(x)^{\mathbb{1}\{d\in A\}}\right) = \lim_{x\to\infty} C^Y\left(F_{Y_1}(x)^{\mathbb{1}\{1\in A\}}, \dots, F_{Y_d}(x)^{\mathbb{1}\{d\in A\}}\right).$$

Let $\{\mathbf{Y}_n = (Y_{n1}, ..., Y_{nd})\}_{n\geq 1}$ be a sequence of independent random vectors and identically distributed with $\mathbf{Y}$. We have

$$\lim_{x\to\infty} C^X\big(F_{X_1}(x)^{\mathbb{1}\{1\in A\}}, ..., F_{X_d}(x)^{\mathbb{1}\{d\in A\}}\big) = \lim_{x\to\infty}\lim_{n\to\infty} P\Big(\bigcap_{j\in A} \max_{1\leq i\leq n} Y_{i,j} \leq nx\Big)$$

$$= \lim_{x\to\infty}\lim_{n\to\infty} \exp\Big\{-n\Big(1-P\Big(\bigcap_{j\in A} Y_{1,j}\leq nx\Big)\Big)\Big\}$$

$$= \lim_{x\to\infty}\lim_{n\to\infty} \exp\Big\{-n\sum_{\emptyset\neq J\subset A}(-1)^{|J|+1}P\Big(\bigcap_{j\in J} Y_{1,j} > nx\Big)\Big\}$$

$$= \lim_{x\to\infty}\lim_{n\to\infty} \exp\Big\{-n\sum_{\emptyset\neq J\subset A}(-1)^{|J|+1}P\Big(\bigcap_{j\in J} Y_{1,j} > nx|Y_{1,i(J)} > nx\Big)\frac{1}{nx}\Big\}$$

$$= \lim_{x\to\infty} \exp\Big\{-\frac{1}{x}\sum_{\emptyset\neq J\subset A}(-1)^{|J|+1}\lambda_U\Big(\min_{j\in J} Y_{1,j}|Y_{1,i(J)}\Big)\Big\}$$

$$= \lim_{x\to\infty}\Big(1-\frac{1}{x}\sum_{\emptyset\neq J\subset A}(-1)^{|J|+1}\lambda_U\Big(\min_{j\in J} Y_{1,j}|Y_{1,i(J)}\Big)\Big) = \lim_{x\to\infty}\Big(1-\sum_{\emptyset\neq J\subset A}(-1)^{|J|+1}P\Big(\bigcap_{j\in J} Y_{1,j} > x\Big)\Big)$$

$$= \lim_{x\to\infty}\Big(1-\Big(1-P\Big(\bigcap_{j\in A} Y_j\leq x\Big)\Big)\Big) = \lim_{x\to\infty} C^Y\big(F_{Y_1}(x)^{\mathbb{1}\{1\in A\}}, ..., F_{Y_d}(x)^{\mathbb{1}\{d\in A\}}\big).\ \square$$

It follows from the previous Proposition that if $\{\widehat{\mathbf{Y}}_n = (\widehat{Y}_{n,1}, ..., \widehat{Y}_{n,d})\}_{n\geq 1}$ is an i.i.d. sequence such that, for some sequences of constants, $\{\mathbf{a}_n = (a_{n1} > 0, ..., a_{nd} > 0)\}_{n\geq 1}$ and $\{\mathbf{b}_n = (b_{n1} > ..., b_{nd})\}_{n\geq 1}$, the vector of componentwise maxima, $\widehat{\mathbf{M}}_n = (\widehat{M}_{n\,1}, ..., \widehat{M}_{n\,d})$, satisfies

$$P(\widehat{\mathbf{M}}_n \leq \mathbf{a}_n\mathbf{x}+\mathbf{b}_n) \equiv P\Big(\bigcap_{j=1}^{d} \widehat{M}_{nj} \leq a_{nj}\,x_j + b_{nj}\Big) \to F_{\widehat{X}}(x_1, ..., x_d), \tag{30}$$

then $\lambda_U(\widehat{Y}_{s:d}|\widehat{Y}_{d-k+1:d}) = \lambda_U(\widehat{X}_{s:d}|\widehat{X}_{d-k+1:d})$. If we relax the independence assumption between the random vectors of the sequence, this equality is no longer valid in the presence of a non unit multivariate extremal index. The upper $s,k$-extremal coefficient of $\mathbf{Y}_n$ may not coincide with the corresponding coefficient of the limiting MEV distribution.

We start by recalling the definition of multivariate extremal index due to Nandagopalan (1990 [18]). A stationary sequence $\{\mathbf{Y}_n\}_{n\geq 1}$, having common distribution $F_{\mathbf{Y}_n} = F_{\widehat{\mathbf{Y}}_n}$, $n\geq 1$, has extremal index $\theta(\boldsymbol{\tau}) \equiv \theta(\tau_1, ..., \tau_d) \in [0,1]$ when, for each $\boldsymbol{\tau} = (\tau_1, ..., \tau_d) \in \mathbb{R}_+^d$, there exists $\{\mathbf{u}_n^{(\boldsymbol{\tau})} = (u_n^{(\tau_1)}, ..., u_n^{(\tau_d)})\}_{n\geq 1}$, satisfying:

(i) $n(1-F_{Y_{n,j}}(u_{nj})) \underset{n\to\infty}{\longrightarrow} \tau_j$, $j = 1, ..., d$

(ii) $P(\widehat{\mathbf{M}}_n \leq \mathbf{u}_n^{(\boldsymbol{\tau})}) \underset{n\to\infty}{\longrightarrow} \gamma(\boldsymbol{\tau})$

(iii) $P(\mathbf{M}_n \leq \mathbf{u}_n^{(\boldsymbol{\tau})}) \underset{n\to\infty}{\longrightarrow} \gamma(\boldsymbol{\tau})^{\theta(\boldsymbol{\tau})}$.

Just as in one dimension, the extremal index is a key parameter relating the extreme value properties of a stationary sequence $\{\mathbf{Y}_n\}_{n\geq 1}$ to those of the i.i.d. associated sequence $\{\widehat{\mathbf{Y}}_n\}_{n\geq 1}$.

If (30) holds and $\{\mathbf{Y}_n\}_{n\geq 1}$ has multivariate extremal index $\theta(\boldsymbol{\tau})$, then

$$P(\mathbf{M}_n \leq \mathbf{a}_n\mathbf{x}+\mathbf{b}_n) \equiv P\Big(\bigcap_{j=1}^{d} M_{nj} \leq a_{nj}\,x_j + b_{nj}\Big) \to F_X(x_1, ..., x_d),$$

and the MEV d.f. in the limit satisfies:

$$F_{\mathbf{X}}(x_1,...,x_d) = F_{\widehat{X}}(x_1,...,x_d)^{\theta(\tau_1(x_1),...,\tau_d(x_d))}$$

and

$$F_{X_j}(x_j) = F_{\widehat{X}_j}^{\theta_j}(x_j), \tag{31}$$

with

$$\tau_j(x_j) = -\log F_{\widehat{X}_j}(x_j),\ j=1,...,d,$$

and

$$\theta_j = \lim_{\substack{\tau_i\to 0\\ i\neq j}} \theta(\tau_1,...,\tau_d).$$

Any sub-vector $\mathbf{X}_A$ of $\mathbf{X}$ with indices in $A$ also satisfy

$$F_{\mathbf{X}_A}(\mathbf{x}_A) = F_{\widehat{\mathbf{X}}_A}^{\theta_A(\boldsymbol{\tau}(\mathbf{x})_A)}(\mathbf{x}_A), \tag{32}$$

where

$$\theta_A(\boldsymbol{\tau}_A) = \lim_{\substack{\tau_i\to 0\\ i\notin A}} \theta(\tau_1,...,\tau_d).$$

Therefore, the sequence $\{(\mathbf{Y}_n)_A\}_{n\geq 1}$ of sub-vectors with indices in $A$ also has multivariate extremal index $\theta_A(\boldsymbol{\tau}_A)$.

The multivariate extremal index, although dependent of $\tau$, satisfies the following property:

$$\theta(c\tau_1,...,c\tau_d) = \theta(\tau_1,...,\tau_d),\ \forall c>0$$

(Nandagopalan 1990, [18]), which jointly with (32), allows to relate the copula functions of $\widehat{\mathbf{X}}_A$ and $\mathbf{X}_A$. More precisely, denoting $\theta_A^* = \theta_A((1/\theta_1,...,1/\theta_d)_A)$, we have

$$C_{\mathbf{X}_A}((u,...,u)_A) = C_{\widehat{\mathbf{X}}_A}^{\theta_A^*}((u^{1/\theta_1},...,u^{1/\theta_d})_A) = C_{\widehat{\mathbf{X}}_A}((u^{\theta_A^*/\theta_1},...,u^{\theta_A^*/\theta_d})_A).$$

By applying these relations in the computation of the upper $s,k$-extremal coefficient, we find that, in general, the respective values for $\mathbf{X}$ and $\widehat{\mathbf{X}}$ do not coincide. This is illustrated in the example bellow. First observe that, by taking expression (9) and then (10) of Proposition 2.1, we have

$$\begin{aligned}\lambda_U(X_{s:d}|X_{d-k+1:d}) &= \lim_{t\uparrow 1}\frac{\displaystyle\sum_{0\leq i\leq s-1}\sum_{I\in F_i}\sum_{J\subset\overline{I}}(-1)^{|J|}C_{\widehat{\mathbf{X}}_{I\cup J}}\left(\left(t^{\theta^*_{I\cup J}/\theta_1},...,t^{\theta^*_{I\cup J}/\theta_d}\right)_{I\cup J}\right)}{1-\displaystyle\sum_{0\leq i\leq k-1}\sum_{I\in F_i}\sum_{J\subset I}(-1)^{|J|}C_{\widehat{\mathbf{X}}_{\overline{I}\cup J}}\left(\left(t^{\theta^*_{\overline{I}\cup J}/\theta_1},...,t^{\theta^*_{\overline{I}\cup J}/\theta_d}\right)_{\overline{I}\cup J}\right)}\\ &= \lim_{t\uparrow 1}\frac{\displaystyle\sum_{0\leq i\leq s-1}\sum_{I\in F_i}\sum_{J\subset I}(-1)^{|J|}\overline{C}_{\widehat{\mathbf{X}}_{\overline{I}\cup J}}\left(\left((1-t)^{\theta^*_{\overline{I}\cup J}/\theta_1},...,(1-t)^{\theta^*_{\overline{I}\cup J}/\theta_d}\right)_{\overline{I}\cup J}\right)}{1-\displaystyle\sum_{0\leq i\leq k-1}\sum_{I\in F_i}\sum_{J\subset\overline{I}}(-1)^{|J|}\overline{C}_{\widehat{\mathbf{X}}_{I\cup J}}\left(\left((1-t)^{\theta^*_{I\cup J}/\theta_1},...,(1-t)^{\theta^*_{I\cup J}/\theta_d}\right)_{I\cup J}\right)}.\end{aligned} \tag{33}$$

**Example 4** *Let $\mathbf{Z}=\{Z_n\}_{n\geq 1}$ be an i.i.d. sequence with marginal d.f. $F^*$ in the max-domain of attraction of the extreme value distribution $G^*$. Set $\mathbf{Y}=\{\mathbf{Y}_n=(Y_{n1},Y_{n2},Y_{n3})\}_{n\geq 1}$, where $Y_{n1}=Z_n$, $Y_{n2}=Z_{n+1}$, $Y_{n3}=\max(Z_n,Z_{n+1})$, $n\geq 1$. We have*

$$F_{\mathbf{Y}_n}(x_1,x_2,x_3) = \begin{cases} F^*(x_1)F^*(x_2) & ,\ \max\{x_1,x_2\}\leq x_3\\ F^*(x_1)F^*(x_3) & ,\ x_1\leq x_3<x_2\\ F^*(x_3)F^*(x_2) & ,\ x_2\leq x_3<x_1\\ {F^*}^2(x_3) & ,\ x_3\leq\min\{x_1,x_2\}\end{cases}$$

*and, if* $F^{*^n}(a_n x+b_n)\to G^*(x)$ *for some sequences of constants* $\{a_n>0\}_{n\geq 1}$ *and* $\{b_n\}_{n\geq 1}$*, then*

$$F^n_{\mathbf{Y}_n}(a_nx_1+b_n,a_nx_2+b_n,a_nx_3+b_n)\underset{n\to\infty}{\to}F_{\widehat{X}}(x_1,x_2,x_3)=\begin{cases}G^*(x_1)G^*(x_2) & ,\max\{x_1,x_2\}\leq x_3\\ G^*(x_1)G^*(x_3) & ,x_1\leq x_3<x_2\\ G^*(x_3)G^*(x_2) & ,x_2\leq x_3<x_1\\ G^{*^2}(x_3) & ,x_3\leq\min\{x_1,x_2\}\end{cases}\tag{34}$$

*Moreover,*

$$\begin{aligned}&\lim_{n\to\infty}P(M_{n1}\leq a_nx_1+b_n,M_{n2}\leq a_nx_2+b_n,M_{n3}\leq a_nx_3+b_n)\\ =&\lim_{n\to\infty}F^{*^{n-1}}(a_n\min\{x_1,x_2,x_3\}+b_n)=G^*(\min\{x_1,x_2,x_3\})=\min\{G^*(x_1),G^*(x_2),G^*(x_3)\}\\ \equiv&\ F_{\mathbf{X}}(x_1,x_2,x_3),\end{aligned}\tag{35}$$

*which has totally dependent marginals. In order to compute the extremal index, we have from (34),*

$$\gamma(\tau_1,\tau_2,\tau_3)=\begin{cases}e^{-(\tau_1+\tau_2)} & ,\min\{\tau_1,\tau_2\}\geq\frac{1}{2}\tau_3\\ e^{-(\tau_1+\frac{\tau_3}{2})} & ,\tau_1\geq\frac{1}{2}\tau_3>\tau_2\\ e^{-(\tau_2+\frac{\tau_3}{2})} & ,\tau_2\geq\frac{1}{2}\tau_3>\tau_1\\ e^{-\tau_3} & ,\frac{1}{2}\tau_3\geq\max\{\tau_1,\tau_2\},\end{cases}$$

*which, together with (35), lead us to,*

$$\theta(\tau_1,\tau_2,\tau_3)=\begin{cases}\frac{\tau_1}{\tau_1+\tau_2} & ,\tau_1\geq\tau_2\geq\frac{1}{2}\tau_3\\ \frac{\tau_2}{\tau_1+\tau_2} & ,\tau_2>\tau_1\geq\frac{1}{2}\tau_3\\ \frac{\tau_1}{\tau_1+\frac{\tau_3}{2}} & ,\tau_1\geq\frac{1}{2}\tau_3>\tau_2\\ \frac{\tau_2}{\tau_2+\frac{\tau_3}{2}} & ,\tau_2\geq\frac{1}{2}\tau_3>\tau_1\\ \frac{1}{2} & ,\frac{1}{2}\tau_3\geq\max\{\tau_1,\tau_2\}.\end{cases}$$

*Hence, we have successively,*

$$\theta_{\{1,2\}}(\tau_1,\tau_2)=\begin{cases}\frac{\tau_1}{\tau_1+\tau_2} & ,\tau_1\geq\tau_2\\ \frac{\tau_2}{\tau_1+\tau_2} & ,\tau_2>\tau_1\ ,\end{cases}$$

$$\theta_{\{1,3\}}(\tau_1,\tau_3)=\begin{cases}\frac{\tau_1}{\tau_1+\frac{\tau_3}{2}} & ,\tau_1\geq\frac{1}{2}\tau_3\\ \frac{1}{2} & ,\tau_1<\frac{1}{2}\tau_3\ ,\end{cases}$$

$$\theta_{\{2,3\}}(\tau_2,\tau_3)=\begin{cases}\frac{\tau_2}{\tau_2+\frac{\tau_3}{2}} & ,\tau_2\geq\frac{1}{2}\tau_3\\ \frac{1}{2} & ,\tau_2<\frac{1}{2}\tau_3\ ,\end{cases}$$

$\theta_1=1$, $\theta_2=1$ *and* $\theta_3=1/2$. *We also have,* $\theta^*_{\{1,2,3\}}=\theta(\frac{1}{\theta_1},\frac{1}{\theta_2},\frac{1}{\theta_3})=1/2$, $\theta^*_{\{1,2\}}=\theta_{\{1,2\}}(\frac{1}{\theta_1},\frac{1}{\theta_2})=1/2$, $\theta^*_{\{1,3\}}=\theta_{\{1,3\}}(\frac{1}{\theta_1},\frac{1}{\theta_3})=1/2$, $\theta^*_{\{2,3\}}=\theta_{\{2,3\}}(\frac{1}{\theta_2},\frac{1}{\theta_3})=1/2$.

*Now we apply the first representation in (33) to obtain, for instance,* $\lambda_U(X_{1:3}|X_{3:3})$*, which we already know that must be unit by Proposition 2.6. We have*

$$C_{\widehat{X}}(u_1,u_2,u_3)=\begin{cases}u_1u_2 & ,\max\{u_1,u_2\}\leq u_3\\ u_1u_3 & ,u_1\leq u_3<u_2\\ u_2u_3 & ,u_2\leq u_3<u_1\\ u_3^2 & ,u_3\leq\min\{u_1,u_2\},\end{cases}$$

$$C_{\widehat{X}_{\{1,2\}}}(u_1,u_2)=u_1u_2,$$

$$C_{\widehat{X}_{\{1,3\}}}(u_1,u_3)=\begin{cases} u_1u_3 & ,\ u_1\le u_3 \\ u_3^2 & ,\ u_3<u_1, \end{cases}$$

$$C_{\widehat{X}_{\{2,3\}}}(u_2,u_3)=\begin{cases} u_2u_3 & ,\ u_2\le u_3 \\ u_3^2 & ,\ u_3<u_2, \end{cases}$$

*and hence we obtain*

$$\lambda_U(X_{1:3}|X_{3:3})=\lim_{t\uparrow 1}\frac{1-t-t-t^2+t^{1/2}t^{1/2}+tt^{1/2}+tt^{1/2}-t^{1/2}t^{1/2}}{1-t^{1/2}t^{1/2}}=\lim_{t\uparrow 1}\frac{1-2t-t^2+2t^{3/2}}{1-t}=1.$$

*However, we have*

$$\lambda_U(\widehat{X}_{1:3}|\widehat{X}_{3:3})=\lim_{t\uparrow 1}\frac{1-t-t-t^2+t^2+t^2+t^2-t^2}{1-t^2}=\lim_{t\uparrow 1}\frac{1-2t+t^2}{1-t^2}=0\neq\lambda_U(X_{1:3}|X_{3:3}).$$

*Observe that this example also illustrates that Proposition 2.7 is only a sufficient condition, since* $F_{\widehat{\mathbf{X}}}$ *has no independent marginals but* $\lambda_U(\widehat{X}_{1:3}|\widehat{X}_{3:3})=0$. $\square$

By embeding tail dependence of $\mathbf{X}$ in its tail dependence function $\lambda_U^{\mathbf{X}}$, which exists everywhere on $[0,+\infty[^d$ for a multivariate extreme value distribution, we will allow a new reading for the relation between $\lambda_U(X_{s:d}|X_{d-k+1:d})$ and $\lambda_U(\widehat{X}_{s:d}|\widehat{X}_{d-k+1:d})$. We first remark that, for each $\mathbf{x}=(x_1,...,x_d)\in[0,+\infty[^d$,

$$\lambda_U^{\mathbf{X}_A}(\mathbf{x}_A)=\lim_{t\downarrow 0}\frac{P\left(\bigcap_{i\in A}F_{X_i}(X_i)>(1-t)^{x_i}\right)}{t}=\lim_{t\uparrow 1}\frac{P\left(\bigcap_{i\in A}F_{X_i}(X_i)>t^{x_i}\right)}{1-t}.$$

Moreover, $\lambda_U^{\mathbf{X}_A}$ satisfies the homogeneity propertie

$$\lambda_U^{\mathbf{X}_A}(\alpha\mathbf{x}_A)=\lim_{t\downarrow 0}\frac{P\left(\bigcap_{i\in A}F_{X_i}(X_i)>(1-t)^{\alpha x_i}\right)}{t}=\lim_{t\downarrow 0}\frac{P\left(\bigcap_{i\in A}F_{X_i}(X_i)>(1-t)^{x_i}\right)}{\frac{t}{\alpha}}=\alpha\lambda_U^{\mathbf{X}_A}(\mathbf{x}_A), \tag{36}$$

for each $\alpha>0$.
Accordingly the Proposition 2.9, $\lambda_U(X_{s:d}|X_{d-k+1:d})$ and $\lambda_U(\widehat{X}_{s:d}|\widehat{X}_{d-k+1:d})$ can be computed from its tail dependence functions $\lambda_U^X$ and $\lambda_U^{\widehat{X}}$, respectively. Next result gives us a better insight about the extreme values of $\widehat{\mathbf{X}}$ which are taken into account for the value of $\lambda_U(X_{s:d}|X_{d-k+1:d})$.

**Proposition 3.4** *Let* $\{\widehat{\mathbf{Y}}_n\}_{n\geq 1}$ *be an i.i.d. sequence with distribution* $F_{\widehat{\mathbf{Y}}}$ *in the domain of attraction of the multivariate extreme value distribution* $F_{\widehat{\mathbf{X}}}$ *and* $\{\mathbf{Y}_n\}_{n\geq 1}$ *a stationary sequence with common distribution function* $F_{\mathbf{Y}_n}=F_{\widehat{\mathbf{Y}}_n}$ *and multivariate extremal index* $\theta(\tau_1,...,\tau_d)$, $(\tau_1,...,\tau_d)\in\mathbb{R}_+^d$. *Then, for the multivariate extreme limiting distribution,*

$$F_{\mathbf{X}}(x_1,...,x_d)=\lim_{n\to\infty}P(\mathbf{M}_n\leq\mathbf{a}_n\mathbf{x}+\mathbf{b}_n),$$

*where* $\{\mathbf{a}_n>0\}_{n\geq 1}$ *and* $\{\mathbf{b}_n\}_{n\geq 1}$ *are some sequences of constants, we have*

$$\lambda_U(X_{s:d}|X_{d-k+1:d})=\frac{\displaystyle\sum_{0\leq i\leq s-1}\sum_{I\in F_i}\sum_{J\subset I}(-1)^{|J|}\theta^*_{\overline{I}\cup J}\lambda_U^{\widehat{\mathbf{X}}_{\overline{I}\cup J}}\left(\left(\tfrac{1}{\theta_1},...,\tfrac{1}{\theta_d}\right)_{\overline{I}\cup J}\right)}{\displaystyle\sum_{\emptyset\neq J\subset\{1,...,d\}}(-1)^{|J|}\theta^*_J\lambda_U^{\widehat{\mathbf{X}}_J}\left(\left(\tfrac{1}{\theta_1},...,\tfrac{1}{\theta_d}\right)_J\right)-\sum_{1\leq i\leq k-1}\sum_{I\in F_i}\sum_{J\subset\overline{I}}(-1)^{|J|}\theta^*_{I\cup J}\lambda_U^{\widehat{\mathbf{X}}_{I\cup J}}\left(\left(\tfrac{1}{\theta_1},...,\tfrac{1}{\theta_d}\right)_{I\cup J}\right)},$$

*provided the ratio is defined.*

**Proof.** From representation (33), we have

$$\lambda_U(X_{s:d}|X_{d-k+1:d}) = \lim_{t\uparrow 1} \frac{\sum\limits_{0\le i\le s-1}\sum\limits_{I\in F_i}\sum\limits_{J\subset I}(-1)^{|J|}\frac{1}{1-t}\,\overline{C}_{\hat{\mathbf{x}}_{\overline{I}\cup J}}\Big(\big((1-t)^{\theta^*_{\overline{I}\cup J}/\theta_1},\dots,(1-t)^{\theta^*_{\overline{I}\cup J}/\theta_d}\big)_{\overline{I}\cup J}\Big)}{\frac{1}{1-t}-\sum\limits_{0\le i\le k-1}\sum\limits_{I\in F_i}\sum\limits_{J\subset \overline{I}}(-1)^{|J|}\frac{1}{1-t}\,\overline{C}_{\hat{\mathbf{x}}_{I\cup J}}\Big(\big((1-t)^{\theta^*_{I\cup J}/\theta_1},\dots,(1-t)^{\theta^*_{I\cup J}/\theta_d}\big)_{I\cup J}\Big)}$$

$$= \frac{\sum\limits_{0\le i\le s-1}\sum\limits_{I\in F_i}\sum\limits_{J\subset I}(-1)^{|J|}\lambda_U^{\hat{\mathbf{x}}_{\overline{I}\cup J}}\Big(\theta^*_{\overline{I}\cup J}\big(\tfrac{1}{\theta_1},\dots,\tfrac{1}{\theta_d}\big)_{\overline{I}\cup J}\Big)}{\sum\limits_{\emptyset\ne J\subset\{1,\dots,d\}}(-1)^{|J|}\lambda_U^{\hat{\mathbf{x}}_J}\Big(\theta^*_J\big(\tfrac{1}{\theta_1},\dots,\tfrac{1}{\theta_d}\big)_J\Big)-\sum\limits_{1\le i\le k-1}\sum\limits_{I\in F_i}\sum\limits_{J\subset\overline{I}}(-1)^{|J|}\lambda_U^{\hat{\mathbf{x}}_{I\cup J}}\Big(\theta^*_{I\cup J}\big(\tfrac{1}{\theta_1},\dots,\tfrac{1}{\theta_d}\big)_{I\cup J}\Big)}.$$

Now, just apply (36) to obtain the result. □

The distribution of a $d$-dimensional vector $\mathbf{Y} = (Y_1, ..., Y_d)$ is min-stable (Joe 1997, [8]) if, for each $s > 0$, there exist $\mathbf{a}_s = (a_{s1}, ..., a_{sd})$ and $\mathbf{b}_s = (b_{s1} > 0, ..., b_{sd} > 0)$ such that for each $\mathbf{x} = (x_1, ..., x_d)$,

$$P^s\big(\cap_{j=1}^d Y_j > x_j\big) = P\big(\cap_{j=1}^d Y_j > a_{sj} + b_{sj}x_j\big). \tag{37}$$

For a min-stable distribution $F_{\mathbf{Y}}$, with identical distributed marginals, there exist a constant $\overline{\epsilon}(\mathbf{Y})$ such that

$$P\big(\cap_{j=1}^d Y_j > x\big) = P^{\overline{\epsilon}(\mathbf{Y})}(Y_1 > x),\ \forall x \in \mathbb{R}.$$

In fact, for each $x \in \mathbb{R}$ such that $P(\cap_{j=1}^d Y_j > x) > 0$, we have

$$P\big(\cap_{j=1}^d Y_j > x\big) = P^{\overline{\epsilon}(x)}(Y_1 > x),\ \text{with } \overline{\epsilon}(x) \ge 1.$$

From (37), for each $s > 0$ there exists $a_s$ and $b_s > 0$ constants such that $\overline{\epsilon}(x) = \overline{\epsilon}(a_s + b_s x)$. Therefore $\overline{\epsilon}(x)$ must be a constant which we shall denote by $\overline{\epsilon}(\mathbf{Y})$.
If $\mathbf{Y}$ (or $F_{\mathbf{Y}}$) is positive upper orthant dependent then $\overline{\epsilon}(\mathbf{Y}_A) \le |A|$, for each $A \subset \{1, ..., d\}$.

We compute now $\lambda_L(Y_{d-k+1:d}|Y_{s:d})$ from the coefficients $\overline{\epsilon}(\mathbf{Y}_A)$, presenting in this way a counterpart of Proposition 3.1 for min-stable distributions.

**Proposition 3.5** *If* $\mathbf{Y}$ *has min-stable distribution then, for any* $1 \le s < d-k+1 \le d$*, we have*

$$\lambda_L(Y_{d-k+1:d}|Y_{s:d}) = \frac{\sum_{\emptyset\ne J\subset\{1,\dots,d\}}(-1)^{|J|}\overline{\epsilon}(\mathbf{Y}_J) + \sum_{i=1}^{k-1}\sum_{I\in f_I}\sum_{J\subset\overline{I}}(-1)^{|J|}\overline{\epsilon}(\mathbf{Y}_{I\cup J})}{-\sum_{i=0}^{s-1}\sum_{I\in F_i}\sum_{J\subset I}(-1)^{|J|}\overline{\epsilon}(\mathbf{Y}_{\overline{I}\cup J})},$$

*provided the ratio is defined.*

**Proof.** If we take (12) in Proposition 2.1, we have

$$\lambda_L(Y_{d-k+1:d}|Y_{s:d}) =$$

$$= \lim_{t\downarrow 0}\frac{1+\sum\limits_{\emptyset\ne J\subset\{1,\dots,d\}}(-1)^{|J|}P\Big(\bigcap\limits_{j\in J}F_{Y_j}(Y_j)>t\Big)+\sum\limits_{i=1}^{k-1}\sum\limits_{I\in F_i}\sum\limits_{J\subset\overline{I}}(-1)^{|J|}P\Big(\bigcap\limits_{j\in I\cup J}F_{Y_j}(Y_j)>t\Big)}{1-\sum\limits_{i=0}^{s-1}\sum\limits_{I\in F_i}\sum\limits_{J\subset I}(-1)^{|J|}P\Big(\bigcap\limits_{j\in\overline{I}\cup J}F_{Y_j}(Y_j)>t\Big)}$$

$$= \lim_{t\downarrow 0}\frac{1+\sum\limits_{\emptyset\ne J\subset\{1,\dots,d\}}(-1)^{|J|}(1-t)^{\overline{\epsilon}(\mathbf{Y}_J)}+\sum\limits_{i=1}^{k-1}\sum\limits_{I\in F_i}\sum\limits_{J\subset\overline{I}}(-1)^{|J|}(1-t)^{\overline{\epsilon}(\mathbf{Y}_{I\cup J})}}{1-\sum\limits_{i=0}^{s-1}\sum\limits_{I\in F_i}\sum\limits_{J\subset I}(-1)^{|J|}(1-t)^{\overline{\epsilon}(\mathbf{Y}_{\overline{I}\cup J})}}$$

and the result is straightforward by the l'Hospital's rule. □

A multivariate extreme value distribution is min-stable for multivariate minima, and therefore the vector of componentwise minima has the same distribution up to location-scale changes. If the random vectors $\mathbf{X}^{(j)} = (X_1^{(j)}, ..., X_d^{(j)})$, $1 \leq j \leq n$, are independent and identically distributed as $\mathbf{X}$ with MEV distribution, then by taking $\mathbf{Y} = \left(\min_{1\leq j\leq n} X_1^{(j)}, ..., \min_{1\leq j\leq n} X_d^{(j)}\right)$ we have a random vector with min-stable distribution and positive upper orthant dependence, therefore in the conditions of Proposition 3.5.

We remark that all the formulas presented in this paper can be easily calculated computationally and a directory of extremal coefficients could be implemented. Hand made calculations are only endurable for vectors of small dimension.

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