Regular variation, paretian distributions, and the interplay of light and heavy tails in the fractality of asymptotic models

Dinis D. Pestana[1], Sandra M. Aleixo[2] and J. Leonel Rocha[3]

[1] FCUL, Universidade de Lisboa, and CEAUL, Lisboa, Portugal
Email: dinis.pestana@fc.ul.pt

[2] Mathematics Unit, DEC, Instituto Superior de Engenharia de Lisboa, and CEAUL, Lisboa, Portugal
Email: sandra.aleixo@dec.isel.ipl.pt

[3] Mathematics Unit, DEQ, Instituto Superior de Engenharia de Lisboa, and CEAUL, Lisboa, Portugal
Email: jrocha@deq.isel.ipl.pt

Abstract: Classical central limit theorems, culminating in the theory of infinite divisibility, accurately describe the behaviour of stochastic phenomena with asymptotically negligible components. The classical theory fails when a single component may assume an extreme protagonism. The early developments of the speculation theory didn't incorporate the pioneer work of Pareto on heavy tailed models, and the proper setup to conciliate regularity and abrupt changes, in a wide range of natural phenomena, is Karamata's concept of regular variation and the role it plays in the theory of domains of attraction, [8], and Resnick's tail equivalence leading to the importance of generalized Pareto distribution is the scope of extreme value theory, [13].

Waliszewski and Konarski discussed the applicability of the Gompertz curve and its fractal behaviour for instance in modeling healthy and neoplasic cells tissue growth, [15]. Gompertz function is the Gumbel extreme value model, whose broad domain of attraction contains intermediate tail weight laws with a wide range of behaviour.

Aleixo et al. investigated fractality associated with $Beta^*(p,q)$ models, [1], [2], [10] and [11], some of which are generalized Pareto, that span the possible regular variation of tails. We extend the investigation to other extreme stable models, namely Fréchet's and Weibull's types in the General Extreme Value (GEV) model.

Keywords: power laws, extension of the logistic parabola, Gompertz growth model and Gumbel law, extreme value laws, growth models.

1. Introduction

Power laws have a natural connection with fractals and chaotic dynamics, as explained in detail in [14]. But on the other hand, power laws have an important smoothing effect, since power laws like behaviour is at the root of Karamata's regular variation, [8], a theory that became an important tool in many branches of Mathematics, [5]. Since Feller, [6], observed that Doeblin's and Gnedenko theory of the domains of attraction of the additive stable laws could be simply rephrased in Karamata's framework, slow variation and regular variation became the election tool in characterizing domains of attraction, letting De Haan, [7], achieve a complete characterization of domains of attraction for stable extreme value laws, Bingham solve a more general problem of domains of attraction of generalized convolution algebras, [4], and Kozubowski and Rachev characterize geometric-stable domains of attraction, [9]. No one, as far as we are aware, pointed out that weak convergence of types problems in Probability theory are connected to the renormalization group theory in Mathematical Physics, and that the appropriateness of the regular variation theory to deal with domains of attraction is a side-effect of the self-similarity resulting from power laws and scaling.

In here we shall try to bring together similar ideas that, having been developed in diverse fields, seem so far to be strangers to each other. Namely, we shall use an extension of the beta function and of beta densities to develop an extension of Verhults growth model leading to the Gompertz function — which, in fact, plays an important role in extreme value theory, where it is known as Gumbel distribution function.

The Gumbel extreme stable law has in its domain of attraction laws both with infinite right endpoint and with finite right endpoint. In fact, laws in the Gumbel model domain of attraction aren't heavy tailed, and this can be a serious drawback for the Gompertz growth model proposed by Waliszewski and Konarski as a cancer growth model, [15]. A simple extension of the differential equation whose solution is the Gompertz function leads to the Fréchet extreme value model, which for some values of the malthusian parameter can be very heavy-tailed.

Observe that the introduction of a generalized Pareto densities family, first described by Pickands, [12], is at the core of a presentation of extreme value theory in a POT (Peaks over Thresholds) setting; the paretian laws are the stable elements in this approach, and once again quasi-power functions are substantive in characterizing domains of attraction, and Resnick's tail equivalence is the key concept tying the two theories, [13].

In section 2, we introduce a family of probability density functions tied to the classical beta family. In fact, a *Beta*(p,2) probability density function can be viewed as the truncation of the MacLaurin series expansion for the models *Beta**(p,2) we investigate. We focus our interest in the *Beta**(2,2) model, constructing an extension of the Verhulst model, that show that stable laws for maxima provide a natural framework to study growth when some external behaviour is expected as in the case of cancer growth.

In section 3, we present a bare sketch of the dynamical behaviour of the models based on *Beta**(p,q) laws, focusing once again on the $p=q=2$ case, due to space restrictions. Finaly, in section 4, we point out how these models are connected to generalized mixtures of power functions, a fact that seems to be at the core of the fractal and ultimately chaotic behaviour of the iterates of the difference growth model for non-overlaping generations.

2. An extension of the Beta densities, and Gompertz growth model

The Verhulst growth model has a prestigious history in population dynamics, and the developments of fractal geometry in the XXth Century enlightened some curious features, such has chaotic behaviour followed by extinction, when the malthusian reproduction rate *r* is "large" in the logistic parabola

$$f(x)=r\,x(1-x)\ \mathrm{I}_{(0,1)}(x).$$

Observing that $f(x)=r\,x(1-x)\mathrm{I}_{(0,1)}(x)=f_{r,2,2}(x)$ is proportional to the *Beta(2,2)* density, the authors investigated, [1], [2], [3], [10], and [11], the dynamical behaviour of functions

$$f_{r,p,q}(x)=r\,x^{p}\,(1-x)^{q-1}\ \mathrm{I}_{(0,1)}(x)$$

proportional to $Beta(p,q)$ densities, discovering namely that interesting chaotic patterns of behaviour were closely related to the generalized skewness and kurtosis functions — i.e., to the standardized third and fourth central moments of the model, [3].
In here we use a non-trival extension of Euler's Beta function

$$\mathrm{B}(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x,\quad p,q>0,$$

namely

$$\mathrm{B}^*(p,q)=\int_0^1 x^{p-1}(-\ln x)^{q-1}\mathrm{d}x,\quad p,q>0,$$

to extend the Verhults growth model.
Observe that $-\ln x=\sum_{k=1}^{\infty}\frac{(1-x)^k}{k}$ when $x\approx 0$, and hence $\mathrm{B}(p,q)$ may be viewed has a first order approximation of $\mathrm{B}^*(p,q)$.

Note that, while $\mathrm{B}(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, where $\Gamma(\alpha)=\int_0^{\infty}x^{\alpha-1}\mathrm{e}^{-x}\mathrm{d}x,\alpha>0$, is the Euler's gamma function, $\mathrm{B}^*(p,q)=\frac{\Gamma(q)}{p^q}$. In particular,

$$\mathrm{B}^*(p,2)=\sum_{k=1}^{\infty}\frac{\mathrm{B}(p,k+1)}{k}\frac{1}{p^2}.$$

Hence

$$h_{p,q}(x)=\frac{1}{Beta^*(p,q)}x^{p-1}(-\ln x))^{q-1}\,\mathrm{I}_{(0,1)}(x)=\frac{p^q}{\Gamma(q)}x^{p-1}(-\ln x))^{q-1}\,\mathrm{I}_{(0,1)}(x)$$

is a new probability density function, which we designated by $Beta^*(p,q)$ density, and in particular, the $Beta^*(2,2)$ density

$$h_{2,2}(x)=4\,x\,(-\ln x)\,\mathrm{I}_{(0,1)}(x)$$

is a mixture of the $Beta(2,k+1)$ densities, with weights

$$w_k=\frac{4}{k(k+1)(k+2)}.$$

Now, we consider a natural extention of the logistic parabola growth model inspired in this new $h_{2,2}$ probabilistic model, namely

$$G'(x)=r\,G(x)(-\ln G(x)),\; r>0,\; G(x)\in(0,1)$$

whose solution $G(x)=\mathrm{e}^{\mathrm{e}^{-rx+C}}$, C constante, is the Gompertz growth model. A similar observation is made in [17]. Observe, also, that $\Lambda(x)=\mathrm{e}^{\mathrm{e}^{-rx}}\mathrm{I}_{\mathrm{IR}}(x)$ is the widely used extreme value Gumbel model.
Further observe that the solution of

$$G'_\nu(x)=r\,x^\nu\,G_\nu(x)(-\ln G_\nu(x)),\, r>0,\; G_\nu(x)\in(0,1)$$

is $G_\nu(x)=\mathrm{e}^{\mathrm{e}^{-\frac{rx^{\nu+1}}{\nu+1}+C}}$, C constante.

Hence, when $\nu\to-1$, with $C=\dfrac{r}{\nu+1}$, using L'Hôpital rule we get

$$\lim_{\nu\to-1}\frac{-r\left(x^{\nu+1}-1\right)}{\nu+1}=-r\ln x$$

and we obtain the solution $G_{-1}(x)=\mathrm{e}^{-x^r}$ which is a Fréchet-r extreme value distribution. Laws in the extreme value Fréchet domain of attraction for maxima must have infinte right endpoint, and can be severely heavy-tailed.

3. Dynamical study of the models proportional to $Beta^*(p,q)$ densities — a brief sketch

Consider the family of unimodal maps with a parameter $r\in[m,M]$, $m\in\mathrm{IR}_0^+$, where $g_{r,p,q}:]0,1]\to[0,1]$, such that

$$g_{r,p,q}(x) = r\, x^{p-1}(1-x)^{q-1}$$

satisfies the following conditions:

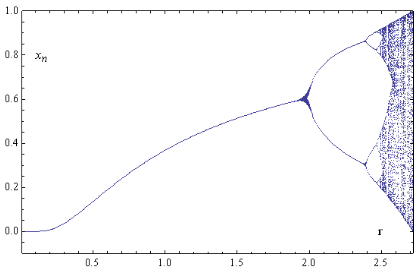
- $g_{r,p,q} \in C^3(\,]0,1]\,)$;
- $g'_{r,p,q}(x) \neq 0, \forall\, x \neq c$ (c is the critical point of $g_{r,p,q}$);
- $g'_{r,p,q}(c) = 0$ and $g''_{r,p,q}(c) < 0$, meaning that $g_{r,p,q}$ is strikly increasing in $]0,c[$ and strikly decreasing in $]c,1]$;
- $g_{r,p,q}(0^+) = g_{r,p,q}(1) = 0$;
- $g_{m,p,q}(c) = 0$ and $g_{M,p,q}(c) = 1$;
- $\forall\, x \neq c \wedge x \neq x_{nd} \wedge x > x_{ps}$, with $0 < x_{ps} < x_{nd}$, $S(g_{r,p,q}(x)) < 0$ where
    - $S(g_{r,p,q}(x)) = \dfrac{g'''_{r,p,q}(x)}{g'_{r,p,q}(x)} - \dfrac{3}{2}\left(\dfrac{g''_{r,p,q}(x)}{g'_{r,p,q}(x)}\right)^2$ represents the Schwarz derivative of $g_{r,p,q}(x)$. Note that $S(g_{r,p,q}(c)) = -\infty$;
    - the domain of the Schwarz derivative of $g_{r,p,q}(x)$ is given by $x \in\,]0,1] \setminus \{x_{nd}\}$;
    - $x_{ps} \in\,]0, x_{nd}[$ is such that $S(g_{r,p,q}(x)) \geq 0$ for $x \in\,]0, x_{ps}]$ (as this condition is verified at the beginning of the interval $]0,1]$, it does not disturb the dynamical behaviour of $g_{r,p,q}$).

The Schwarz derivative of $g_{r,p,q}(x)$ is given by

$$S(g_{r,p,q}(x)) = -\frac{q\,(q-1)^2(q-2) + 4\,(p-1)\,q\,\left(q^2 - 3q + 2\right)\ln x + \left(6\,p^2 - 12\,p + 5\right)(q-1)^2 \ln^2 x}{2\,x^2\, \ln^2 x\,(q - 1 + (p-1)\ln x)^2}$$
$$- \frac{2\left(2\,p^3 - 6\,p^2 + 5\,p - 1\right)(q-1)\ln^3 x + p\,(p-2)(p-1)^2 \ln^4 x}{2\,x^2\, \ln^2 x\,(q - 1 + (p-1)\ln x)^2}.$$

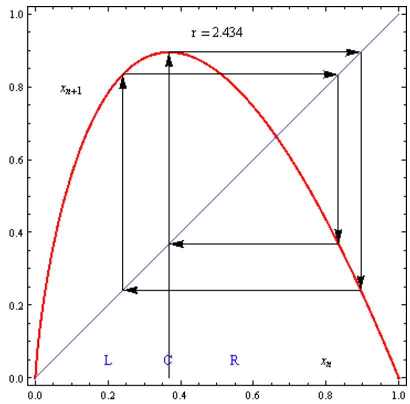
Note that these maps $g_{r,p,q}(x)$ are proportional to the probability density function $h_{p,q}(x)$.

In Fig. 1 below, we exemplify the bifurcation diagrams corresponding to the unimodal family $g_{r,p,q}$, using the case $p = q = 2$. For different values of $p, q > 0$, diagrams are in essence similar, with some non-utterly dramatic changes in shape and scale. In Fig. 2 we can see a 4-periodic orbit for the map proportional to the $Beta^*(2,2)$ model.



Bifurcation Diagram for $g_{r,2,2}$

Figure 1



4-period orbit for $g_{r,2,2}$

Figure 2

Period	Sequence	$Beta^*(2,3)$	$Beta^*(2,2)$	$Beta^*(3,2)$	h_{top}
2	$(CR)^\infty$	1.230	2.218	4.412	0
4	$(CRLR)^\infty$	1.481	2.434	4.678	0
8	$(CRLR^3LR)^\infty$	1.535	2.478	4.736	0
6	$(CRLR^3)^\infty$	1.607	2.535	4.812	0.241
8	$(CRLR^5)^\infty$	1.643	2.564	4.847	0.304
7	$(CRLR^4)^\infty$	1.689	2.599	4.886	0.382
5	$(CRLR^2)^\infty$	1.722	2.624	4.927	0.414
7	$(CRLR^2LR)^\infty$	1.749	2.645	4.967	0.442
8	$(CRLR^2LR^2)^\infty$	1.769	2.660	4.998	0.468
3	$(CRL)^\infty$	1.786	2.673	5.038	0.481
	CRL^∞	1.847	2.718	5.437	ln 2

Table 1: Values of the parameter *r* of some symbolic sequences and topologycal entropy.

In Table 1, we can observe the values of the parameter r for which are obtained the several periodic orbits correspondent to the respective kneading symbolic sequences. This values were obtained from the construction of symbolic lists for increasing values of the parameter r, using suitable computacional programs implemented in *Mathematica* 7.0. As a consequence of all this maps $g_{r,p,q}$ are unimodal, the admissible sequences are exactly the same, as well as the respective values of topologycal entropy, cf. [1] and [2].

4. $Beta^*(p,q)$ models as signed mixtures of power functions

In section 2, we have seen that the probability density function $h_{2,2}$ is a convex mixture of the $Beta(2,k+1)$ densities. On the other hand, using the binomial expansion of the factor $(1-x)^k$, we observe that each $Beta(2,k+1)$ probability density function is a signed mixture — in the sense that the sum of the weights is 1,

$$\sum_{j=0}^{k} w_j^* = \frac{1}{\mathrm{B}(2,k+1)} \sum_{j=0}^{k} (-1)^j \frac{k!}{(j+2)\, j!\,(k-j)!} = 1,$$

although odd indexed weights are negative and even indexed weights are positive — of $Beta(j+2,1)$ densities:

$$\frac{x\,(1-x)^k}{\mathrm{B}(2,k+1)} = \sum_{j=0}^{k} \frac{(-1)^j}{j+2} \binom{k}{j} \frac{1}{\mathrm{B}(2,k+1)} \,(j+2) x^{j+1}$$

Therefore, the $Beta^*(2,2)$ density function $h_{2,2}(x) = 4\,x\,(-\ln x)\,\mathrm{I}_{(0,1)}(x)$ is a signed mixture

$$4\,x\,(-\ln x)\,\mathrm{I}_{(0,1)}(x) = \sum_{k=1}^{\infty}\sum_{j=0}^{k} (-1)^j \frac{4\,k!}{(j+2)\, j!\,(k-j)!}\,(j+2)\,x^{j+1}\,\mathrm{I}_{(0,1)}(x)$$

of $Beta(j+2,1)$ densities, in other words a signed mixture of power laws with natural exponents, with $\sum_{k=1}^{\infty}\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{4}{k\,(j+2)} = 1$.

This is at the core of the duality observed in the growth of populations — for instance populations of neoplasic cells in tumours — as modeled by regularly varying tailed distributions and the correponding asymptotic extreme value laws: each positive even component influencing the exponential growth rate is moderated by the retroaction of the next negative odd term. The relevance of power laws in fractality and chaos is well-known [14], and the connection with regular variation is evident. Our more versatile family of model inherits and amplifies the interplay between malthusian growth term and retroactive term of the logistic parabola, which results in a interesring heavy and light tail equilibrium result.

Acknowledgements

This research has been partially supported by FCT/OE, CEAUL and ISEL.

References

[1] S. M. Aleixo, J. L. Rocha and D. D. Pestana. Populational growth models proportional to beta densities with Allee effect. *Am. Inst. of Phys.*, 1124: 3-12, 2008.

[2] S. M. Aleixo, J. L. Rocha and D. D. Pestana. Dynamical behaviour in the parameter space: new populational growth models proportional to beta densities. In Luzar-Stiffler, V., Jarec, I. and Bekic, Z. (eds.), *Proc. of the ITI 2009, 31th Int. Conf. on Information Technology Interfaces (ITI 2009)*, 213-218, 2009.

[3] S. M. Aleixo, J. L. Rocha and D. D. Pestana. Dynamical analysisin probabilistic models: random Beta(p,2)-Cantor sets, *Notas e Comunicações do CEAUL*, 2010.

[4] N. H. Bingham. Factorisation theory and domains of attraction for generalised convolution algebras, *Proc. London Mathem. Soc*. (3) 23: 16-30, 1971.

[5] N. H. Bingham, C. M. Goldie and J. L.Teugels. *Regular Variation*, Cambridge University Press, Cambridge, 1987.

[6] W. Feller. On regular variation and local limit theorems, in J. Neyman (ed.) *Proc. Fifth Berkeley Symp. Math. Stat. Prob*., vol II, Part I: 373-388, 1967.

[7] L. De Haan. *On Regular Variation and its Applications to the Weak Convergence os Sample Extremes*, Math. Centre Tract 32, Amsterdam, 1970.

[8] J. Karamata. Sur un mode de croissance régulière des fonctions, *Mathematica* (Cluj), 4: 38-53, 1930.

[9] T. J. Kozubowski and S. T.Rachev. Univariate geometric stable distributions, *J. Computational Analysis Appl*., 1: 177-217, 1999.

[10] D. D. Pestana, S. M. Aleixo and J. L. Rocha. Hausdorff dimension of the random middle third Cantor set. In Luzar-Stiffler, V., Jarec, I. and Bekic, Z. (eds.), *Proceedings of the ITI 2009, 31th International Conference on Information Technology Interfaces*, p. 279-284, 2009a.

[11] D. D. Pestana, S. M. Aleixo and J. L. Rocha. The Beta(p,1) extensions of the random (uniform) Cantor sets. *Disc. Math. Prob. Stat.*, 29 (in press), 2009b.

[12] J. Pickands. Statistical inference using extreme order statistics, *Ann. Satatist*., 3: 119-131, 1975.

[13] S. I. Resnick. Tail equivalence and its applications. *J. Appl. Prob*. 8: 136-156, 1971.

[14] M. Schroeder. *Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise*, Dover, New York, 2009.

[15] J. Waliszewski and J. Konarski. A Mystery of the Gompertz Function, in G.A. Losa, D. Merlini, T. F .Nonnenmacher and E.R. Weibel (eds.), *Fractals in Biology and Medicine*, Birkhäuser, Basel, 277-286, 2005.