Uniformity

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Abstract Transformations such as $V = X + Y - I[X+Y]$ or $W = \min\left(\frac{X}{Y}, \frac{1-X}{1-Y}\right)$ can be used to augment uniform random samples, and Sukhatme's transformation can be used to augment uniform order statistics. We discuss the bearing of these facts in testing uniformity, an important issue in the field of combining p-values in meta analytical syntheses.

1 Introduction

Let us assume that the p-values $\{p_k\}_{k=1}^n$ are known from testing H_{0k} vs. H_{Ak}, $k = 1, \ldots, n$, in n independent studies on some common issue, and our aim is to achieve a decision on the overall question H_0^* : all the H_{0k} are true *vs.* H_A^* : some of the H_{Ak} are true. As there are many different ways in which H_0^* can be false, selecting an appropriate test is in general unfeasible. On the other hand, combining the available p_k's so that $T(p_1, \ldots, p_n)$ is the observed value of a random variable whose sampling distribution under H_0^* is known is a simple issue, since under H_0^*, $p = (p_1, \ldots, p_n)$ is the observed value of a random sample $P = (P_1, \ldots P_n)$ from a

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standard uniform population. In fact, several different sensible combined testing procedures are often used (Pestana, 2009).

Therefore an important issue is to test wether a given sequence $\{p_k\}_{k=1}^n$ is or isn't a sample from a standard uniform population. Recently Paul (2003) discussed new characterizations of the uniform, but as they are formulated in terms of expected values, those didn't lead directly to new simple tests of uniformity. Gomes *et al.* (2009) exploited the possibility of using computationally augmented samples to test uniformity, with the surprising result that power can decrease with sample augmentation in the class of alternatives they used. Sequeira (2009) explains why this is so, and in section 2 below we further discuss the question. In the present paper we use Sukhatme's transformation to suggest new ways of dealing with the matter.

Sukhatme's (1937) transformation, from which Rényi's representation of exponential order statistics can easily be derived, appears in David and Nagaraja (2003, p. 17-18) and in Johnson *et al.* (1995, p. 305), with slightly different presentations, applied to the study of exponential and of uniform order statistics, respectively. Durbin (1961) used ordered spacings of the uniform to investigate the construction of exact tests. In section 3 we use a Sukhatme's like transformation to augment the set of order statistics from a uniform parent, and in Section 4 we investigate power issues when they are used in testing uniformity.

2 Uniformity vs. mixtures of Uniform and Beta(1,2)

Gomes *et al.* (2009) introduced the family $\{X_m\}_{m\in[-2,2]}$ of absolutely continuous random variables, with probability density function $f_{X_m}(x) = \left(mx - \frac{m-2}{2}\right) I_{(0,1)}(x)$, (the uniform density corresponds to $m = 0$; for $m \in (0,2]$, X_m is a convex mixture of Beta(1,1) and Beta(2,1), and for $m \in [-2,0]$, X_m is a mixture of Beta(1,1) and Beta(1,2)). Observe that for all $m \in [-2,0)$, $\mathbb{P}[X_m \le x] - \mathbb{P}[U \le x] = \frac{m}{2}\, x(x-1) > 0$ for all $x \in (0,1)$, and thus pseudo-random numbers generated by X_m tend to be closer to 0 than pseudo-random numbers generated by a standard uniform random variable U. Thus this family can give important hints on non-uniformity of the set of p-values, cf. the concepts of random p-values in Kulinskaya *et al.* (2008) and of generalized p-values in Hartung *et al.* (2008).

Moreover, the inverse of the corresponding distribution function is

$$F_{X_m}^{-1}(y) = \frac{\frac{m}{2} - 1 + \sqrt{\left(\frac{m}{2} - 1\right)^2 + 2my}}{m}$$

and the generation of pseudo-random numbers from X_m for simulation studies is therefore straightforward.

Let U and X be two independent standard uniform random variables. The random variables $V = U + X - I[U+X]$, where $I[x]$ denotes the largest integer not greater than x, and $W = \min\left(\frac{U}{X}, \frac{1-U}{1-X}\right)$ are uniform and independent of X (Deng and George, 1992). This fact was used by Gomes *et al.* (2009) for computationally augmenting samples and to assess the power of detecting non-uniformity when the sample comes in fact from X_m, $m \in [-2,0]$, with the strange result that power doesn't improve for increased samples.

The explanation is however simple: $\min\left(\frac{X_m}{X_p}, \frac{1-X_m}{1-X_p}\right) \stackrel{d}{=} X_{\frac{mp}{6}}$. Hence in case the algorithm uses uniform pseudo-random numbers to augment the sample, the augmented slice will in fact be a uniform sub-sample, and power decreases. Brilhante *et al.* (2010) present better results using left-skewed parent pseudo-random numbers.

Still, the use of the family $\{X_m\}_{m\in[-2,2]}$ has many advantages, and instead of augmenting the sample *externally*, as in the above mentioned papers, by using $V_m = U + X_m - I[U+X]$ and $W_m = \min\left(\frac{U}{X_m}, \frac{1-U}{1-X_m}\right)$, with the spurious effect of always generating uniform pseudo p-values, we can use an alternative approach when the purpose is to test the null hypothesis of uniformity *vs.* X_m parent:

- choose at random one $p_j \in \{p_k\}_{k=1}^n$.
- generate $n-1$ pseudo-p's of the form $\min\left(\frac{p_j}{p_k}, \frac{1-p_j}{1-p_k}\right)$, $k \neq j$.

3 Order statistics, spacings and an extension of Sukhatme's transformation

Let $X = (X_1, X_2, \ldots, X_n)$ be a random sample from the absolutely continuous positive random variable X with probability density function f_X, and $(X_{1:n}, X_{2:n}, \ldots, X_{n:n})$ the corresponding vector of ascending order statistics. For convenience we assume that left-endpoint $\alpha_X = 0$ and we define $X_{0:n} = \alpha_X = 0$.

The joint probability density function of the spacings $S_k = X_{k:n} - X_{k-1:n}$, $k = 1, \ldots, n$, is

$$f_{(S_1,S_2,\ldots,S_n)}(s_1, s_2, \ldots, s_n) = n!\, f_{(X_1,X_2,\ldots,X_n)}(s_1, s_1+s_2, \ldots, s_1+\cdots+s_n)$$

($s_k > 0$, $k = 1, \ldots, n$, and if the right-endpoint ω_X is finite, $\sum_{k=1}^n s_k < \omega_X$; in this case we can consider the rightmost spacing $S_{n+1} = \omega_X - X_{n:n}$, but this can be expressed as a function $\omega_X - \sum_{k=1}^n S_k$). Hence the joint probability density function of the ascending reordering of those n spacings is

$$f_{(S_{1:n},S_{2:n},\ldots,S_{n:n})}(y_1, y_2, \ldots, y_n) = (n!)^2 f_{(X_1,X_2,\ldots,X_n)}(y_1, y_1+y_2, \ldots, y_1+\cdots+y_n),$$

where $0 < y_1 < y_2 < \ldots < y_n$ and $\sum_{k=1}^n y_k < \omega_X$.

Now define

$$W_k = (n+1-k)(S_{k:n} - S_{k-1:n}), \quad k = 1, \ldots, n,$$

(similar to Sukhatme's transformation, as defined in David and Nagaraja, 2003, but applied to ascendingly ordered spacings) again with the convention $S_{0:n} = 0$.

The joint probability density function of $(W_1, W_2, \ldots, W_n)$ is

$$f_{(W_1,W_2,\ldots,W_n)}(w_1, w_2, \ldots, w_n) = n!\, f_{(X_1,X_2,\ldots,X_n)}\left(\tfrac{w_1}{n}, \tfrac{2w_1}{n} + \tfrac{w_2}{n-1}, \ldots, w_1 + \cdots + w_n\right),$$

$w_k > 0$, $k = 1, \ldots, n$, (observe that the k-th argument is

$$\frac{kw_1}{n} + \frac{(k-1)w_2}{n-1} + \cdots + \frac{(k+1-j)w_j}{n+1-j} + \cdots + \frac{w_k}{n+1-k},\ k = 1, \ldots, n),$$

and the joint probability density function of the vector of partial sums $Y_k = \sum_{j=1}^{k} W_j$, $k = 1, \ldots, n$, is

$$f_{(Y_1,Y_2,\ldots,Y_n)}(y_1, y_2, \ldots, y_n) = n!\, f_{(X_1,X_2,\ldots,X_n)}\left(\tfrac{y_1}{n}, \ldots, \sum_{j=1}^{k} \tfrac{(k+1-j)(y_j - y_{j-1})}{n+1-j}, \ldots, y_n\right)$$

with $0 < y_1 < y_2 < \ldots < y_n$ and the convention $y_0 = 0$.

If $X \frown \text{Uniform}(0, \omega_X)$, then

$$f_{(X_1,X_2,\ldots,X_n)}\left(\tfrac{y_1}{n}, \ldots, \sum_{j=1}^{k} \tfrac{(k+1-j)(y_j - y_{j-1})}{n+1-j}, \ldots, y_n\right) = \frac{1}{\omega_X^n} = f_{(X_1,X_2,\ldots,X_n)}(y_1, y_2, \ldots, y_n),$$

and hence $(Y_1, Y_2, \ldots, Y_n) \stackrel{d}{=} (X_{1:n}, X_{2:n}, \ldots, X_{n:n})$. [1].

This suggests that uniformity can be investigated testing wether $\{X_{k:n}\}_{k=1}^{n}$ and $\{Y_k\}_{k=1}^{n}$ can be considered samples from the same distribution. Unfortunately, under the null hypothesis that the parent distribution is standard uniform,

$$(Y_1, Y_2, \ldots, Y_n) \stackrel{d}{=} (X_{1:n}, X_{2:n}, \ldots, X_{n:n}),$$

but the two vectors are not independent, since we can re-express $Y_k = \sum_{j=1}^{k} S_{j:n} + (n-k)S_{k:n}$, and consequently $Y_n = X_{n:n}$. Thus, Smirnov two-sample test is of no use in the present situation.

However the observation of Fig. 1, where we compare the empirical distribution function (edf) corresponding to the order statistics (red) and the y_k (blue), in case of uniform parent and Beta(1,2) parent, suggests that $D_n^* = \sup_x |F_n^*(x) - G_n^*(x)|$, where F_n^* stands for the order statistics edf and G_n^* for the accumulated y_k edf, will

[1] Observe that if $\omega_X < \infty$, we can consider $n+1$ spacings, with $S_{n+1} = \omega_X - X_{n:n}$; of course in this situation S_{n+1}, $S_{n+1:n+1}$ and W_{n+1} (where in this case it is convenient to use the transformation

$$W_k = (n+2-k)(S_{k:n+1} - S_{k-1:n+1}),$$

as in Johnson *et al.*, 1995, p. 305) can be expressed as simple functions of the predecessor members of the corresponding samples. We still get the result that $(Y_1, Y_2, \ldots, Y_n) \stackrel{d}{=} (X_{1:n}, X_{2:n}, \ldots, X_{n:n})$ in case of standard uniform parent X.

be greater under the alternative $H_A : X$ non-uniform with support (0,1) than under the null hypothesis $H_0 : X \frown \text{Uniform}(0,1)$.

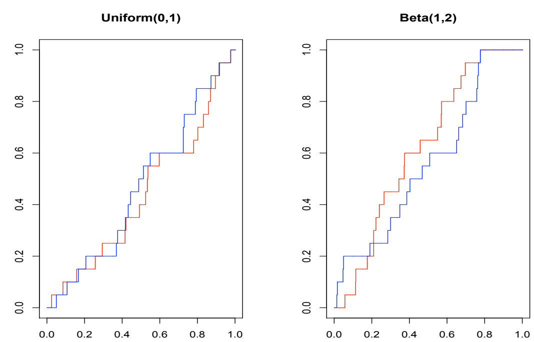

Fig. 1 Empirical distribution functions F^*_{20} and G^*_{20} for Uniform(0,1) and Beta(1,2) parents

For uniformity testing purposes we present in Table 1 the upper critical points of D^*_n, $n = 3(1)30(5)100$, when the underlying parent is standard uniform ($U \overset{\mathrm{d}}{=} X_0$). These points were obtained by generating 10 000 independent replicates of the sample $(D^*_{n,1}, D^*_{n,2}, \ldots, D^*_{n,50})$ and defining the quantile of order p of D^*_n as the mean of the samples quantiles for $p = 0.9, 0.925, 0.95, 0.975, 0.99, 0.995, 0.999$.

We also performed a simulation study of the proportion of rejections of uniformity when the underlying parent was X_m, $m \in [-2,0]$, and when making pairwise comparisons of the order statistics edf and the $\{y_k\}$ edf (the process of generating $\{y_k\}$ was iteratively repeated 10 000 times). Observe that the rationale for this procedure relies on the fact that if the original observations $\{p_k\}$ are indeed uniform, the "Sukhatme's" $\{y_k\}$ would be order statistics of standard uniform, and hence repeating Sukhatme's like generation procedure we would obtain again a set of order statistics of the standard uniform.

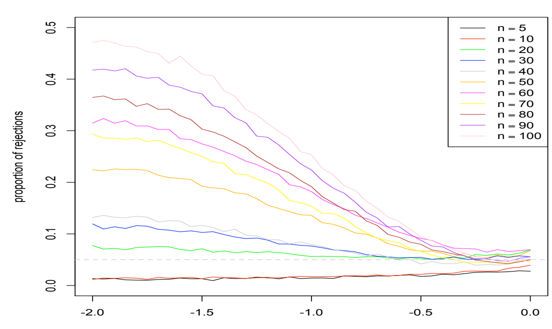

Fig. 2 Proportion of rejections of uniformity at level 0.05 using Sukhatme's like transformation when the underlying parent is X_m, $m \in [-2,0]$

From Fig. 2 we observe that the proportion of rejections of uniformity increases with n. However, the extended Sukhatme's like transformed data performs badly in detecting departures from uniformity when $n < 20$. This situation can obviously constitute a problem when combining p-values in meta analytical syntheses since the number of available (reported) p-values is usually small.

Another way of assessing the usefulness of this extended Sukhatme's transformation in testing uniformity is by calculating the area limited by the edf's F_n^* and G_n^*, since under the validity of the null hypothesis $X \frown \text{Uniform}(0,1)$, the area between the two curves should be zero — bigg area values should indicate a departure from uniformity. In Table 2 we compare the areas obtained by simulation (10 000 runs) for some values of n when the underlying parents are standard uniform and Beta(1,2). Analyzing Table 2 we see that the area is indeed inferior for the standard uniform parent, except for some few cases. However, the differences between the two areas can be very small, which can difficult the task of testing uniformity with this procedure.

4 Conclusion

It seems worth to point out that the entropy of X_m, $m \in [-2,0]$ is

$$H(X_m) = -\int_0^1 f_{X_m}(x)\ln\left(f_{X_m}(x)\right)\mathrm{d}x = 0.5 + \ln(2) + \frac{\ln\left(\left(\frac{2-m}{2+m}\right)^m\right)}{8} - \frac{\ln(4-m^2)}{2} + \frac{\ln\left(\frac{2-m}{2+m}\right)}{2m},$$

cf. Fig. 3 (for a detailed study of entropy, cf. Johnson, 2004), and hence, as

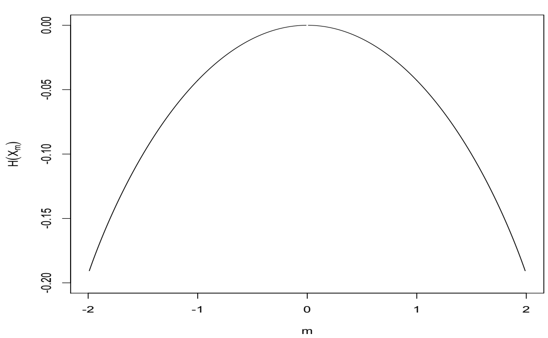

Fig. 3 Entropy of X_m, $m \in [-2,2]$.

$\min\left(\frac{X_m}{X_p}, \frac{1-X_m}{1-X_p}\right) \overset{\mathrm{d}}{=} X_{\frac{mp}{6}}$, its entropy is always nearer the entropy of the standard uniform than the entropy of X_m or X_p. It is therefore no surprising that comparison of the two techniques for testing uniformity — so far in the restricted situation of testing uniformity *vs.* X_m parent — shows that the test suggested by the extended

Table 1 Critical points of D_n^* when the underlying parent is standard uniform[a]

n	**0.9**	**0.925**	**0.95**	**0.975**	**0.99**	**0.995**	**0.999**
3	0.667	0.667	0.667	0.667	0.667	0.667	0.667
4	0.605	0.656	0.703	0.734	0.747	0.747	0.747
5	0.600	0.610	0.634	0.682	0.753	0.753	0.753
6	0.548	0.580	0.620	0.666	0.736	0.736	0.736
7	0.542	0.563	0.589	0.632	0.712	0.712	0.712
8	0.509	0.529	0.558	0.605	0.686	0.686	0.686
9	0.484	0.509	0.540	0.582	0.660	0.660	0.660
10	0.470	0.491	0.518	0.558	0.635	0.635	0.635
11	0.454	0.472	0.498	0.537	0.612	0.612	0.612
12	0.436	0.455	0.482	0.520	0.592	0.592	0.592
13	0.422	0.441	0.466	0.503	0.574	0.574	0.574
14	0.410	0.429	0.452	0.487	0.557	0.557	0.557
15	0.398	0.415	0.438	0.472	0.539	0.539	0.539
16	0.387	0.404	0.427	0.460	0.525	0.525	0.525
17	0.377	0.393	0.416	0.447	0.511	0.511	0.511
18	0.368	0.385	0.406	0.437	0.498	0.498	0.498
19	0.359	0.376	0.396	0.427	0.486	0.486	0.486
20	0.352	0.367	0.387	0.416	0.474	0.474	0.474
21	0.345	0.360	0.379	0.408	0.463	0.463	0.463
22	0.337	0.352	0.371	0.399	0.453	0.453	0.453
23	0.331	0.345	0.363	0.391	0.444	0.444	0.444
24	0.325	0.339	0.357	0.384	0.435	0.435	0.435
25	0.319	0.332	0.350	0.376	0.427	0.427	0.427
26	0.313	0.326	0.344	0.370	0.419	0.419	0.419
27	0.308	0.321	0.338	0.363	0.411	0.411	0.411
28	0.302	0.315	0.332	0.357	0.404	0.404	0.404
29	0.298	0.311	0.327	0.352	0.400	0.400	0.400
30	0.293	0.306	0.322	0.345	0.392	0.392	0.392
35	0.273	0.285	0.300	0.321	0.363	0.363	0.363
40	0.257	0.268	0.282	0.302	0.341	0.341	0.341
45	0.243	0.253	0.267	0.286	0.322	0.322	0.322
50	0.231	0.241	0.254	0.272	0.306	0.306	0.306
55	0.221	0.230	0.242	0.260	0.292	0.292	0.292
60	0.212	0.221	0.232	0.249	0.280	0.280	0.280
65	0.204	0.212	0.224	0.239	0.269	0.269	0.269
70	0.197	0.205	0.216	0.231	0.260	0.260	0.260
75	0.190	0.198	0.209	0.223	0.251	0.251	0.251
80	0.185	0.193	0.202	0.217	0.244	0.244	0.244
85	0.179	0.186	0.196	0.210	0.236	0.236	0.236
90	0.174	0.182	0.191	0.204	0.229	0.229	0.229
95	0.170	0.177	0.186	0.199	0.223	0.223	0.223
100	0.166	0.172	0.181	0.194	0.217	0.217	0.217

[a] The standard errors of the critical points are less than or equal to 0.001.

Sukhatme's transformation is more powerful than the test using augmented samples as described in section 2. The general question of comparing analytically empirical distribution functions of correlated samples remains unsolved, even for simple forms of weak dependence, only simulation results in well defined situations seems feasible.

Table 2 Area limited by the functions F_n^* and G_n^* when the underlying parents are Uniform(0,1) and Beta(1,2)

	Beta(1,2)		**Uniform(0,1)**	
n	**area**	**s.e.**	**area**	**s.e.**
5	0.0366	0.00188	0.0333	0.00179
10	0.0848	0.00279	0.1027	0.00304
15	0.0794	0.00270	0.1216	0.00327
20	0.0860	0.00280	0.0820	0.00274
25	0.0620	0.00241	0.0608	0.00239
30	0.0823	0.00275	0.0495	0.00217
35	0.0699	0.00255	0.0526	0.00223
40	0.0742	0.00262	0.0411	0.00199
45	0.0665	0.00249	0.0450	0.00207
50	0.1005	0.00301	0.0319	0.00176
55	0.0927	0.00290	0.0370	0.00189
60	0.0774	0.00267	0.0376	0.00190
65	0.0830	0.00276	0.0247	0.00155
70	0.0648	0.00246	0.0425	0.00202
75	0.0371	0.00189	0.1369	0.00344
80	0.0682	0.00252	0.0388	0.00193
85	0.0702	0.00256	0.0403	0.00197
90	0.0901	0.00286	0.0395	0.00195
95	0.0701	0.00255	0.0358	0.00186
100	0.0730	0.00260	0.0498	0.00218

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