Combination of multivariate extremal indexes

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Abstract: Let $\mathbf{Y} = (Y_1, \ldots, Y_{p+q})$ be a random vector with multivariate extreme value distribution corresponding to the limit distribution of the maximum of a $(p+q)-$dimensional stationary sequence $\{\mathbf{X}_n = (X_{n,1}, \ldots, X_{n,p+q})\}_{n\geq 1}$ with extremal index. We give necessary and sufficient conditions for the sub-vectors $\mathbf{Y}^{(p)} = (Y_1, \ldots, Y_p)$ and $\mathbf{Y}^{(q)} = (Y_{p+1}, \ldots, Y_{p+q})$ to be independent or totally dependent which involve relations between the extremal indexes of the sequences $\{\mathbf{X}_n = (X_{n,1}, \ldots, X_{n,p+q})\}_{n\geq 1}$, $\{\mathbf{X}_n^{(p)} = (X_{n,1}, \ldots, X_{n,p})\}_{n\geq 1}$ and $\{\mathbf{X}_n^{(q)} = (X_{n,p+1}, \ldots, X_{n,p+q})\}_{n\geq 1}$. We illustrate the main results with an auto-regressive sequence.

Key words: Multivariate extremal index, dependence conditions, multivariate extreme value distribution, total dependence.

1 Introduction

Let $\mathbf{X} = \{\mathbf{X}_n^{(d)} = (X_{n,1}, \ldots, X_{n,d}\}_{n\geq 1}$ be a $d-$dimensional stationary sequence with common distribution function (d.f.) $Q(\mathbf{x}^{(d)}) = Q(x_1, \ldots, x_d)$, $\mathbf{x}^{(d)} \in \mathbb{R}^d$, and $\mathbf{M}_n = (M_{n,1}, \ldots, M_{n,d})$ the vector of pointwise maxima, where $M_{n,i} = \max(X_{1,i}, \ldots, X_{n,i})$ is the maximum of ith component. Denote by $\widehat{\mathbf{M}}_n = (\widehat{M}_{n,1}, \ldots, \widehat{M}_{n,d})$ the corresponding vector of pointwise maxima of the associated d-dimensional sequence, $\widehat{\mathbf{X}} = \{\widehat{\mathbf{X}}_n^{(d)}\}_{n\geq 1}$ of independent and identically distributed (i.i.d.) random vectors having the same distribution function Q.

For each $d > 1$ and $\mathbf{a}^{(d)}, \mathbf{b}^{(d)} \in \mathbb{R}^d$, $\mathbf{a}^{(d)} \leq \mathbf{b}^{(d)}$, if and only if $a_j \leq b_j$, for all $j = 1, 2, \ldots, d$.

If there exist sequences $\{\mathbf{a}_n = (a_{n,1} > 0, \ldots, a_{n,d} > 0)\}_{n\geq 1}$ and $\{\mathbf{b}_n = (b_{n,1}, \ldots, b_{n,d})\}_{n\geq 1}$, such that for $\mathbf{u}(\mathbf{x}^{(d)}) = \{\mathbf{u}_n(\mathbf{x}^{(d)}) = (a_{n,1}x_1 + b_{n,1}, \ldots, a_{n,d}x_d + b_{n,d})\}_{n\geq 1}$,

$$P\left(\widehat{\mathbf{M}}_n \leq \mathbf{u}_n(\mathbf{x}^{(d)})\right) = P\left(\bigcap_{j=1}^{d} \left\{\widehat{M}_{n,j} \leq a_{n,j}x_j + b_{n,j}\right\}\right) \xrightarrow[n\to\infty]{} G(\mathbf{x}^{(d)}),\ \mathbf{x}^{(d)} \in \mathbb{R}^d,$$

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where G is a d.f. with non-degenerate margins, then $Q(\mathbf{x}^{(d)})$ is said to be in the max domain of attraction of G and G is said to be a Multivariate Extreme Value (MEV) distribution function.

We shall assume, without loss of generality, that the univariate marginal distributions of G are equal to F.

It is well known that the relationship between the distribution function $G(\mathbf{x}^{(d)})$, $\mathbf{x}^{(d)} \in \mathbb{R}^d$, and its marginal distributions $F(x_j), j = 1, \ldots, d$, can be characterized by its copula function (or dependence function), D_G, introduced in Sibuya (1960), which exhibits a number of interesting properties (Deheuvels, 1978; Hsing, 1989), namely its stability equation:

$$D_G^t(y_1, \ldots, y_d) = D_G(y_1^t, \ldots, y_d^t), \ \forall t > 0 \text{ and } (y_1, \ldots, y_d) \in [0,1]^d . \tag{1.1}$$

If the stationary sequence $\mathbf{X}$ satisfies some long range dependence conditions, $D(\mathbf{u}_n(\mathbf{x}^{(d)}))$ of Hsing (1989) or $\Delta(\mathbf{u}_n(\mathbf{x}^{(d)}))$ of Nandagopalan (1990), and

$$P\left(\mathbf{M}_n \leq \mathbf{u}_n(\mathbf{x}^{(d)})\right) \underset{n\to\infty}{\longrightarrow} H(\mathbf{x}^{(d)}), \ \mathbf{x}^{(d)} \in \mathbb{R}^d,$$

where H is a d.f. with non-degenerate components, then the d.f. H is also a MEV distribution function.

The MEV distribution functions H and G can be related through the function multivariate extremal index, $\theta(\boldsymbol{\tau}^d) = \theta(\tau_1, \ldots, \tau_d) \in \mathbb{R}_+^d$ introduced by Nandagopalan (1990), which is a measure of the clustering among the extreme values of a multivariate stationary sequence.

Definition 1.1 *A $d-$dimensional stationary sequence $\mathbf{X}$, is said to have multivariate extremal index $\theta^{\mathbf{X}}\left(\boldsymbol{\tau}^{(d)}\right) \in [0,1]$, if for each $\boldsymbol{\tau}^{(d)} = (\tau_1, \ldots, \tau_d) \in \mathbb{R}_+^d$ there exists $\mathbf{u}_n^{(\boldsymbol{\tau}^{(d)})} = (u_{n,1}^{(\tau_1)}, \ldots, u_{n,d}^{(\tau_d)})$, $n \geq 1$, satisfying*

$$nP(X_{1,j} > u_{n,j}^{(\tau_j)}) \underset{n\to\infty}{\longrightarrow} \tau_j, \ j = 1, 2, \ldots, d, \quad P\left(\widehat{\mathbf{M}}_n \leq \mathbf{u}_n^{(\boldsymbol{\tau}^{(d)})}\right) \underset{n\to\infty}{\longrightarrow} G(\boldsymbol{\tau}^{(d)})$$

and

$$P\left(\mathbf{M}_n \leq \mathbf{u}_n^{(\boldsymbol{\tau}^{(d)})}\right) \underset{n\to\infty}{\longrightarrow} G^{\theta^{\mathbf{X}}(\boldsymbol{\tau}^{(d)})}(\boldsymbol{\tau}^{(d)}).$$

Just as in one dimension, the extremal index is the key parameter relating the extreme value properties of a stationary sequence to those of indendent random vectors from the same $d-$dimensional marginal distribution. However, unlike the one dimensional case, it is not a constant for the whole process, but instead depends on the vector $\boldsymbol{\tau}^{(d)}$.

It is now clear that the existence of the multivariate extremal index allow us to write

$$H(\mathbf{x}^{(d)}) = G^{\theta^{\mathbf{X}}(\boldsymbol{\tau}^{(d)}\mathbf{x}^{(d)})}(\mathbf{x}^{(d)})$$

with $\tau_j \equiv \tau_j(x_j) = -\log F(x_j), \ j = 1, 2, \ldots, d$.

Taking $d = p + q$, it follows, as a consequence of the definition of multivariate extremal index, that the sequences $\mathbf{X}^{(p)} = \{\mathbf{X}_n^{(p)} = (X_{n,1}, \ldots, X_{n,p})\}_{n\geq 1}$ and $\mathbf{X}^{(q)} = \{\mathbf{X}_n^{(q)} = (X_{n,p+1}, \ldots, X_{n,p+q})\}_{n\geq 1}$ have, respectively, extremal indexes

$$\theta^{\mathbf{X}^{(p)}}(\boldsymbol{\tau}^{(p)}) = \lim_{\substack{\tau_j \to 0^+ \\ j=p+1,\ldots,p+q}} \theta^{\mathbf{X}}(\boldsymbol{\tau}^{(p+q)})$$

and

$$\theta^{\mathbf{X}^{(q)}}(\boldsymbol{\tau}^{(q)}) = \lim_{\substack{\tau_j \to 0^+ \\ j=1,\ldots,p}} \theta^{\mathbf{X}}(\boldsymbol{\tau}^{(p+q)}).$$

Hereinafter, let $\mathbf{Y} = (Y_1, \ldots, Y_{p+q})$ and $\widehat{\mathbf{Y}} = (\widehat{Y}_1, \ldots, \widehat{Y}_{p+q})$ be two random vectors with distribution functions $G^{\theta^{\mathbf{X}}(\tau^{(p+q)})}(\mathbf{x}^{(p+q)})$ and $G(\mathbf{x}^{(p+q)})$, respectively, $\mathbf{Y}^{(p)} = (Y_1, \ldots, Y_p)$ and $\mathbf{Y}^{(q)} = (Y_{p+1}, \ldots, Y_{p+q})$ be two sub-vectors of $\mathbf{Y}$ and $\widehat{\mathbf{Y}}^{(p)} = (\widehat{Y}_1, \ldots, \widehat{Y}_p)$ and $\widehat{\mathbf{Y}}^{(q)} = (\widehat{Y}_{p+1}, \ldots, \widehat{Y}_{p+q})$ be two sub-vectors of $\widehat{\mathbf{Y}}$.

We give necessary and sufficient conditions for $\mathbf{Y}^{(p)}$ and $\mathbf{Y}^{(q)}$ to be independent or totally dependent by using relations between the extremal indexes $\theta^{\mathbf{X}}(\boldsymbol{\tau}^{(p+q)})$, $\theta^{\mathbf{X}^{(p)}}(\boldsymbol{\tau}^{(p)})$ and $\theta^{\mathbf{X}^{(q)}}(\boldsymbol{\tau}^{(q)})$.

We only consider two sub-vectors $\mathbf{Y}^{(p)}$ and $\mathbf{Y}^{(q)}$ for sake of simplicity, nevertheless the results can be rewriten for several vectors. Considering $p+q$ variables we obtain relations between $\theta^{\mathbf{X}}(\boldsymbol{\tau}^{(p+q)})$ and θ^{X_j}, $j = 1, 2, \ldots, p+q$, as in Martins and Ferreira (2005).

In the notation of the extremal index we shall omit the sequence, whenever it is clear by the context and the argument of the function, that is, we write $\theta(\boldsymbol{\tau}^{(p+q)})$ instead of $\theta^{\mathbf{X}}(\boldsymbol{\tau}^{(p+q)})$, $\theta(\boldsymbol{\tau}^{(p)})$ instead of $\theta^{\mathbf{X}^{(p)}}(\boldsymbol{\tau}^{(p)})$ and $\theta(\boldsymbol{\tau}^{(q)})$ instead of $\theta^{\mathbf{X}^{(q)}}(\boldsymbol{\tau}^{(q)})$.

2 Main results

If the d.f. Q belongs to the domain of attraction of a MEV distribution, G, and $\mathbf{X}$ has extremal index $\theta(\boldsymbol{\tau}^{(p+q)})$, $\boldsymbol{\tau}^{(p+q)} = (\tau_1, \ldots, \tau_{p+q}) \in \mathbb{R}_+^{p+q}$, then we have

$$G^{\theta(\boldsymbol{\tau}^{(p)})}(\mathbf{x}^{(p)})G^{\theta(\boldsymbol{\tau}^{(q)})}(\mathbf{x}^{(q)}) \leq G^{\theta(\boldsymbol{\tau}^{(p+q)})}(\mathbf{x}^{(p+q)}) \leq \min\{G^{\theta(\boldsymbol{\tau}^{(p)})}(\mathbf{x}^{(p)}), G^{\theta(\boldsymbol{\tau}^{(q)})}(\mathbf{x}^{(q)})\}, \tag{2.1}$$

for each $\mathbf{x}^{(p+q)} \in \mathbb{R}^{p+q}$ and $\tau_j = -\log F(x_j)$, $j = 1, \ldots, p+q$.

The inequality on the right holds true for every multivariate distribution. The inequality on the left is a property of MEV distributions. The lower bound corresponds to the case where $\mathbf{Y}^{(p)}$ and $\mathbf{Y}^{(q)}$ are independent and the upper bound corresponds to the case where $\mathbf{Y}^{(p)}$ and $\mathbf{Y}^{(q)}$ are totally dependent.

From (2.1) we obtain the following bounds for the multivariate extremal index function $\theta(\boldsymbol{\tau}^{(p+q)}), \boldsymbol{\tau}^{(p+q)} \in \mathbb{R}_+^{p+q}$.

$$\frac{\max\{\theta(\boldsymbol{\tau}^{(p)})\gamma(\boldsymbol{\tau}^{(p)}), \theta(\boldsymbol{\tau}^{(q)})\gamma(\boldsymbol{\tau}^{(q)})\}}{\gamma(\boldsymbol{\tau}^{(p+q)})} \leq \theta(\boldsymbol{\tau}^{(p+q)}) \leq \frac{\theta(\boldsymbol{\tau}^{(p)})\gamma(\boldsymbol{\tau}^{(p)}) + \theta(\boldsymbol{\tau}^{(q)})\gamma(\boldsymbol{\tau}^{(q)})}{\gamma(\boldsymbol{\tau}^{(p+q)})}, \tag{2.2}$$

where

$$\gamma(\boldsymbol{\tau}^{(p+q)}) = -\log G(F^{-1}(e^{-\tau_1}), \ldots, F^{-1}(e^{-\tau_{p+q}})) = \lim_{n\to\infty} nP\left(\mathbf{X}_n^{(p+q)} \not\leq u_n^{(\boldsymbol{\tau}^{(p+q)})}\right),$$

$$\gamma(\boldsymbol{\tau}^{(p)}) = \lim_{\substack{\tau_j \to 0^+ \\ j=p+1,\ldots,p+q}} \gamma(\boldsymbol{\tau}^{(p+q)}) \text{ and } \gamma(\boldsymbol{\tau}^{(q)}) = \lim_{\substack{\tau_j \to 0^+ \\ j=1,\ldots,p}} \gamma(\boldsymbol{\tau}^{(p+q)}),$$

$\tau_j = -\log F(x_j), j = 1, \ldots, p+q$.

The next proposition, which establishes necessary and sufficient conditions for $\mathbf{Y}^{(p)}$ and $\mathbf{Y}^{(q)}$ to be independent or totally dependent, follows from (2.2).

Proposition 2.1 *Suppose that the common d.f. of* $\mathbf{X} = \{\mathbf{X}_n^{(p+q)}\}_{n\geq 1}$ *belongs to the domain of attraction of a MEV distribution* G *and* $\mathbf{X}$ *has extremal index* $\theta(\boldsymbol{\tau}^{(p+q)})$, $\boldsymbol{\tau}^{(p+q)} \in \mathbb{R}_+^{p+q}$. *Let* $\mathbf{Y} = (Y_1, \ldots, Y_{p+q})$ *and* $\widehat{\mathbf{Y}} = (\widehat{Y}_1, \ldots, \widehat{Y}_{p+q})$ *be two random vectors with d.f.* $G^\theta(\mathbf{x}^{(p+q)})$ *and* $G(\mathbf{x}^{(p+q)})$*, respectively. Consider* $\mathbf{Y}^{(p)} = (Y_1, \ldots, Y_p)$ *and* $\mathbf{Y}^{(q)} = (Y_{p+1}, \ldots, Y_{p+q})$ *be two sub-vectors of* $\mathbf{Y}$ *and* $\widehat{\mathbf{Y}}^{(p)} = (\widehat{Y}_1, \ldots, \widehat{Y}_p)$ *and* $\widehat{\mathbf{Y}}^{(q)} = (\widehat{Y}_{p+1}, \ldots, \widehat{Y}_{p+q})$ *be two sub-vectors of* $\widehat{\mathbf{Y}}$.

(i) *If* $\widehat{\mathbf{Y}}^{(p)}$ *and* $\widehat{\mathbf{Y}}^{(q)}$ *are independent, then* $\mathbf{Y}^{(p)}$ *and* $\mathbf{Y}^{(q)}$ *are independent if and only if*

$$\theta(\boldsymbol{\tau}^{(p+q)}) = \frac{\theta(\boldsymbol{\tau}^{(p)})\gamma(\boldsymbol{\tau}^{(p)}) + \theta(\boldsymbol{\tau}^{(q)})\gamma(\boldsymbol{\tau}^{(q)})}{\gamma(\boldsymbol{\tau}^{(p)}) + \gamma(\boldsymbol{\tau}^{(q)})}, \ \boldsymbol{\tau}^{(p+q)} \in \mathbb{R}_+^{p+q}. \tag{2.3}$$

(ii) *If* $\widehat{\mathbf{Y}}^{(p)}$ *and* $\widehat{\mathbf{Y}}^{(q)}$ *are totally dependent, then* $\mathbf{Y}^{(p)}$ *and* $\mathbf{Y}^{(q)}$ *are totally dependent if and only if*

$$\theta(\boldsymbol{\tau}^{(p+q)}) = \frac{\max\{\theta(\boldsymbol{\tau}^{(p)})\gamma(\boldsymbol{\tau}^{(p)}), \theta(\boldsymbol{\tau}^{(q)})\gamma(\boldsymbol{\tau}^{(q)})\}}{\max\{\gamma(\boldsymbol{\tau}^{(p)}), \gamma(\boldsymbol{\tau}^{(q)})\}}, \ \boldsymbol{\tau}^{(p+q)} \in \mathbb{R}^{p+q}.$$

The necessary and sufficient conditions for $\mathbf{Y}^{(p)}$ and $\mathbf{Y}^{(q)}$ to be independent or totally dependent given in the previous result demand the evaluation of the extremal index function $\theta(\boldsymbol{\tau}^{(p+q)})$, in each point $\boldsymbol{\tau}^{(p+q)} \in \mathbb{R}_+^{p+q}$. Nevertheless this task can be simplified with the caracterizations, given by Ferreira (2007), for the independence and total dependence of the multivariate marginals of a MEV distribution, which we next recall. This result is a generalization to vectors of Takahashi's result (1988).

Proposition 2.2 *Let* $\mathbf{Y} = (Y_1, \ldots, Y_{p+q})$ *with MEV distribution* $H_{\mathbf{Y}}$, $\mathbf{Y}^{(p)} = (Y_1, \ldots, Y_p)$ *and* $\mathbf{Y}^{(q)} = (Y_{p+1}, \ldots, Y_{p+q})$.

(i) *We have*

$$H_{\mathbf{Y}}(\mathbf{x}^{(p+q)}) = H_{\mathbf{Y}^{(p)}}(\mathbf{x}^{(p)})H_{\mathbf{Y}^{(q)}}(\mathbf{x}^{(q)}), \ \textit{for all } \mathbf{x}^{(p+q)} \in \mathbb{R}^{p+q},$$

if and only if there exists $\mathbf{y}^{(p+q)} \in \mathbb{R}^{p+q}$ *such that*

$$0 < H_{\mathbf{Y}^{(p)}}(\mathbf{y}^{(p)}) < 1, 0 < H_{\mathbf{Y}^{(q)}}(\mathbf{y}^{(q)}) < 1 \ \ e \ \ H_{\mathbf{Y}}(\mathbf{y}^{(p+q)}) = H_{\mathbf{Y}^{(p)}}(\mathbf{y}^{(p)})H_{\mathbf{Y}^{(q)}}(\mathbf{y}^{(q)}).$$

(ii)

1. *If there exists* $\mathbf{y}^{(p+q)} \in \mathbb{R}^{p+q}$ *such that*

$$H_{\mathbf{Y}}(\mathbf{y}^{(p+q)}) = H_{Y_1}(y_1) = \ldots = H_{Y_{p+q}}(y_{p+q}) = a \in]0,1[$$

then, for each two sub-vectors $\mathbf{Y}^{(s)}$ *and* $\mathbf{Y}^{(t)}$ *of* $\mathbf{Y}$*, with* $s + t = p + q$*, it holds*

$$H_{\mathbf{Y}}(\mathbf{x}^{(p+q)}) = \min\{H_{\mathbf{Y}^{(s)}}(\mathbf{x}^{(s)}), H_{\mathbf{Y}^{(t)}}(\mathbf{x}^{(t)})\}, \ \textit{for all } \mathbf{x}^{(p+q)} \in \mathbb{R}^{p+q}.$$

2. *If*

$$H_{\mathbf{Y}}(\mathbf{x}^{(p+q)}) = \min\{H_{\mathbf{Y}^{(p)}}(\mathbf{x}^{(p)}), H_{\mathbf{Y}^{(q)}}(\mathbf{x}^{(q)})\}, \ \textit{for all } \mathbf{x}^{(p+q)} \in \mathbb{R}^{p+q},$$

then there exists $\mathbf{y}^{(p+q)} \in \mathbb{R}^{p+q}$ *such that*

$$H_{\mathbf{Y}}(\mathbf{y}^{(p+q)}) = H_{\mathbf{Y}^{(p)}}(\mathbf{y}^{(p)}) = H_{\mathbf{Y}^{(q)}}(\mathbf{y}^{(q)}) = H_{Y_1}(y_1) = \ldots = H_{Y_{p+q}}(y_{p+q}) = a \in]0,1[.$$

We can now obtain the following propositions which establish that we only need to evaluate the extremal index in some points to conclude the independence or total dependence of $\mathbf{Y}^{(p)}$ and $\mathbf{Y}^{(q)}$.

Proposition 2.3 *Suppose that $Q \in D(G)$ and the sequence $\mathbf{X} = \{\mathbf{X}_n^{(p+q)}\}_{n\geq 1}$ has extremal index $\theta(\boldsymbol{\tau}^{(p+q)})$, $\boldsymbol{\tau}^{(p+q)} \in \mathbb{R}_+^{p+q}$. Let $\mathbf{Y} = (Y_1, \ldots, Y_{p+q})$ be a random vector with d.f. $G^{\theta^{\mathbf{X}}(\boldsymbol{\tau}^{(p+q)})}(\mathbf{x}^{(p+q)})$, $\mathbf{Y}^{(p)} = (Y_1, \ldots, Y_p)$ and $\mathbf{Y}^{(q)} = (Y_{p+1}, \ldots, Y_{p+q})$.*
$\mathbf{Y}^{(p)}$ and $\mathbf{Y}^{(q)}$ are independent if and only if

$$\theta(\mathbf{1}^{(p+q)}) = \frac{\theta(\mathbf{1}^{(p)})\gamma(\mathbf{1}^{(p)}) + \theta(\mathbf{1}^{(q)})\gamma(\mathbf{1}^{(q)})}{\gamma(\mathbf{1}^{(p+q)})}, \tag{2.4}$$

where $\mathbf{1}^{(k)} = (1, \ldots, 1), k > 1$, denotes the k-dimensional unitary vector.

Proof: If $\mathbf{Y}^{(p)}$ and $\mathbf{Y}^{(q)}$ are independent and since (2.2) holds for all $\boldsymbol{\tau}^{(p+q)} \in \mathbb{R}_+^{p+q}$, we have in particular for $\boldsymbol{\tau}^{(p+q)} = (\tau, \ldots, \tau) \in \mathbb{R}_+^{p+q}$, with $\tau \equiv \tau(x) = -\log F(x),\ x \in \mathbb{R}$,

$$\theta(\boldsymbol{\tau}^{(p+q)}) = \frac{\theta(\boldsymbol{\tau}^{(p)})\gamma(\boldsymbol{\tau}^{(p)}) + \theta(\boldsymbol{\tau}^{(q)})\gamma(\boldsymbol{\tau}^{(q)})}{\gamma(\boldsymbol{\tau}^{(p+q)})}.$$

Noticing that $\theta(c\boldsymbol{\tau}^{(k)}) = \theta(\boldsymbol{\tau}^{(k)})$ for each $\boldsymbol{\tau}^{(k)} \in \mathbb{R}_+^k$, $k > 1$ and $c > 0$, we obtain

$$\theta(\boldsymbol{\tau}^{(p+q)}) = \theta(\mathbf{1}^{(p+q)}),\ \theta(\boldsymbol{\tau}^{(p)}) = \theta(\mathbf{1}^{(p)}),\ \theta(\boldsymbol{\tau}^{(q)}) = \theta(\mathbf{1}^{(q)}),$$

and from (1.1), for all $\boldsymbol{\tau}^{(p+q)} = (\tau, \ldots, \tau) \in \mathbb{R}_+^{p+q}$,

$$\begin{aligned}
\gamma(\boldsymbol{\tau}^{(p+q)}) &= -\log G(F^{-1}(e^{-\tau}), \ldots, F^{-1}(e^{-\tau})) \\
&= -\log D_G(e^{-\tau}, \ldots, e^{-\tau}) \\
&= -\log D_G^{\tau}(e^{-1}, \ldots, e^{-1}) \\
&= \tau\gamma(\mathbf{1}^{(p+q)}),
\end{aligned}$$

$\gamma(\boldsymbol{\tau}^{(p)}) = \tau\gamma(\mathbf{1}^{(p)})$ and $\gamma(\boldsymbol{\tau}^{(q)}) = \tau\gamma(\mathbf{1}^{(q)})$. Equality (2.4) is now straightforward.

On the other hand if the relation (2.4) is verified, then for $\mathbf{x}^{(p+q)} = (x, x, \ldots, x)$ we have

$$\begin{aligned}
G_{\mathbf{Y}}(\mathbf{x}^{(p+q)}) &= G^{\theta(\mathbf{1}^{(p+q)})}(\mathbf{x}^{(p+q)}) \\
&= D_G^{\theta(\mathbf{1}^{(p+q)})}(e^{-\tau}, \ldots, e^{-\tau}) \\
&= D_G^{\theta(\mathbf{1}^{(p+q)})\tau}(e^{-1}, \ldots, e^{-1}) \\
&= \exp(-\tau\gamma(\mathbf{1}^{(p+q)})\theta(\mathbf{1}^{(p+q)})) \\
&= \exp(-\tau(\theta(\mathbf{1}^{(p)})\gamma(\mathbf{1}^{(p)}) + \theta(\mathbf{1}^{(q)})\gamma(\mathbf{1}^{(q)}))) \\
&= \exp(-\tau\theta(\mathbf{1}^{(p)})\gamma(\mathbf{1}^{(p)}))\exp(-\tau\theta(\mathbf{1}^{(q)})\gamma(\mathbf{1}^{(q)})) \\
&= G_{\mathbf{Y}^{(p)}}(\mathbf{x}^{(p)})G_{\mathbf{Y}^{(q)}}(\mathbf{x}^{(q)})
\end{aligned}$$

and from Proposition 2.2 we conclude that $\mathbf{Y}^{(p)}$ and $\mathbf{Y}^{(q)}$ are independent. $\square$

Proposition 2.4 *Suppose that* $Q \in D(G)$ $\mathbf{X} = \{\mathbf{X}_n^{(p+q)}\}_{n\geq 1}$ *has extremal index* $\theta(\boldsymbol{\tau}^{(p+q)})$, $\boldsymbol{\tau}^{(p+q)} \in \mathbb{R}_+^{p+q}$. *Let* $\mathbf{Y} = (Y_1, \ldots, Y_{p+q})$ *be a random vector with d.f.* $G^{\theta^{\mathbf{X}}(\boldsymbol{\tau}^{(p+q)})}(\mathbf{x}^{(p+q)})$, $\mathbf{Y}^{(p)} = (Y_1, \ldots, Y_p)$ *and* $\mathbf{Y}^{(q)} = (Y_{p+1}, \ldots, Y_{p+q})$.

(i) *If* $\mathbf{Y}^{(p)}$ *and* $\mathbf{Y}^{(q)}$ *are totally dependent then there exists* $\boldsymbol{\tau}^{(p+q)} \in \mathbb{R}_+^{p+q}$ *with* $\tau_j \equiv \tau_j(x_j) = -\log F(x_j)$, $x_j \in \mathbb{R},\ j = 1, ..., p+q$, *such that*

$$\gamma(\boldsymbol{\tau}^{(p)})\theta(\boldsymbol{\tau}^{(p)}) = \gamma(\boldsymbol{\tau}^{(q)})\theta(\boldsymbol{\tau}^{(q)}) = \theta_1\tau_1 \ldots = \theta_{p+q}\tau_{p+q} = d > 0$$

and

$$\theta(\boldsymbol{\tau}^{(p+q)}) = \frac{1}{\gamma(\frac{\boldsymbol{\tau}^{(p+q)}}{d})}.$$

(ii) *If there exists* $\boldsymbol{\tau}^{(p+q)} \in \mathbb{R}_+^{p+q}$ *with* $\tau_j \equiv \tau_j(x_j) = -\log F(x_j), x_j \in \mathbb{R},\ j = 1, ..., p+q$, *such that*

$$\gamma(\boldsymbol{\tau}^{(p+q)})\theta(\boldsymbol{\tau}^{(p+q)}) = \theta_1\tau_1 \ldots = \theta_{p+q}\tau_{p+q} = d > 0,$$

then $\mathbf{Y}^{(p)}$ *and* $\mathbf{Y}^{(q)}$ *are totally dependent.*

Proof: **(i)** From Proposition 2.2, if $\mathbf{Y}^{(p)}$ and $\mathbf{Y}^{(q)}$ are totally dependent, then there exists $\boldsymbol{\tau}^{(p+q)} \in \mathbb{R}_+^{p+q}$ such that

$$\theta(\boldsymbol{\tau}^{(p)})\gamma(\boldsymbol{\tau}^{(p)}) = \theta(\boldsymbol{\tau}^{(q)})\gamma(\boldsymbol{\tau}^{(q)}) = \theta(\boldsymbol{\tau}^{(p+q)})\gamma(\boldsymbol{\tau}^{(p+q)}) = d = \theta_1\tau_1 = \ldots = \theta_{p+q}\tau_{p+q},$$

with $d \in]0,1[$. Hence

$$\begin{aligned}
\theta(\boldsymbol{\tau}^{(p+q)}) &= \frac{d}{\gamma(\boldsymbol{\tau}^{(p+q)})} = \frac{d}{-\log D_G(\exp(-\tau_1), \ldots, \exp(-\tau_{p+q}))} \\
&= \frac{1}{-\log D_G^{\frac{1}{d}}(\exp(-\tau_1), \ldots, \exp(-\tau_{p+q}))} \\
&= \frac{1}{-\log D_G\left(\exp\left(-\frac{\tau_1}{d}\right), \ldots, \exp\left(-\frac{\tau_{p+q}}{d}\right)\right)} \\
&= \frac{1}{\gamma(\frac{\boldsymbol{\tau}^{(p+q)}}{d})}.
\end{aligned}$$

$\square$

3 Example

The following example illustrates the previous results.

Example 3.1 Let $\{Y_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with common d.f. F and consider an auto-regressive sequence of maxima $\{X_n\}_{n\geq 1}$ defined by

$$X_n = \max\{Y_n, Y_{n+1}\},\ n \geq 1,$$

with marginal distribution F^2.

Let $\left\{u_n^{(\tau_i)}\right\}_{n\geq 1}$, $i = 1, 2, \ldots, p$, and $\left\{v_n^{(\tau_i')}\right\}_{n\geq 1}$, $i = p+1, \ldots, p+q$, be sequences of real numbers such that

$$n(1 - F^2(u_n^{(\tau_i)})) \xrightarrow[n\to\infty]{} \tau_i, \quad i = 1, 2, \ldots, p,$$

and

$$nF^2(-v_n^{(\tau_i')})) \xrightarrow[n\to\infty]{} \tau_i', \quad i = p+1, \ldots, p+q.$$

The sequences $\{X_n\}_{n\geq 1}$ and $\{-X_n\}_{n\geq 1}$ satisfy the local dependence conditions $D''(u_n^{(\tau_i)})$, $i = 1, \ldots, p$, and $D''(v_n^{(\tau_i')})$, $i = p+1, \ldots, p+q$, of Leadbetter and Nandagopalan (1989),

$$nP(X_1 \leq u_n^{(\tau_i)} < X_2) \xrightarrow[n\to\infty]{} \frac{\tau_i}{2},\ i = 1, 2, \ldots, p,$$

and

$$nP(-X_1 \leq v_n^{(\tau_i')} < -X_2) \xrightarrow[n\to\infty]{} \tau_i',\ i = p+1, \ldots, p+q.$$

Hence, the extremal indexes are, respectively,

$$\theta_1 = \lim_{n\to\infty} \frac{nP(X_1 \leq u_n^{(\tau_i)} < X_2)}{nP(X_1 > u_n^{(\tau_i)})} = \frac{1}{2}$$

and

$$\theta_2 = \lim_{n\to\infty} \frac{nP(-X_1 \leq v_n^{(\tau_i')} < -X_2)}{nP(-X_1 > v_n^{(\tau_i')})} = 1.$$

For the sequences

$$\mathbf{X}_n^{(p+q)} = \begin{cases} X_{n,i} = X_n & , i = 1, \ldots p \\ X_{n,i} = -X_n & , i = p+1, \ldots, p+q \end{cases}, \quad \mathbf{X}_n^{(p)} = (X_n, \ldots, X_n) \quad \text{e} \quad \mathbf{X}_n^{(q)} = (-X_n, \ldots, -X_n).$$

we then have

$$\begin{aligned}
P\left(\mathbf{M}_n^{(p)} \leq (u_n^{(\tau_1)}, \ldots, u_n^{(\tau_p)})\right) &= P\left(\max_{1\leq i\leq n} X_i \leq \min_{1\leq j\leq p} u_n^{(\tau_j)}\right) \xrightarrow[n\to\infty]{} \exp\left(-\frac{1}{2}\max_{1\leq j\leq p} \tau_j\right), \\
P\left(\widehat{\mathbf{M}}_n^{(p)} \leq (u_n^{(\tau_1)}, \ldots, u_n^{(\tau_p)})\right) &= P\left(\max_{1\leq i\leq n} \widehat{X}_i \leq \min_{1\leq j\leq p} u_n^{(\tau_j)}\right) \xrightarrow[n\to\infty]{} \exp\left(-\max_{1\leq j\leq p} \tau_j\right), \\
P\left(\mathbf{M}_n^{(q)} \leq (v_n^{(\tau_{p+1}')}, \ldots, v_n^{(\tau_{p+q}')})\right) &= P\left(\max_{1\leq i\leq n} (-X_i) \leq \min_{p+1\leq j\leq p+q} v_n^{(\tau_j')}\right) \xrightarrow[n\to\infty]{} \exp\left(-\max_{p+1\leq j\leq p+q} \tau_j'\right).
\end{aligned}$$

Since the order statistics maximum and minimum are asymptotically independent (Davis (1979, 1982), Pereira and Ferreira (2001), Pereira(2002)) we obtain

$$
\begin{aligned}
& P(\mathbf{M}_n \leq (u_n^{(\tau_1)}, \ldots, u_n^{(\tau_p)}, v_n^{(\tau'_{p+1})}, \ldots, v_n^{(\tau'_{p+q})})) \\
= & \quad P\left(\max_{1\leq i\leq n} X_i \leq \min_{1\leq j\leq p} u_n^{(\tau_j)}, \max_{1\leq i\leq n} \{-X_i\} \leq \min_{p+1\leq j\leq p+q} v_n^{(\tau'_j)} \right) \\
\xrightarrow[n\to\infty]{} & \quad \exp\left(-\frac{1}{2} \max_{1\leq j\leq p} \tau_j \right) \exp\left(- \max_{p+1\leq j\leq p+q} \tau'_j \right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\gamma(\tau^{(p+q)}) &= \gamma(\tau^{(p)}) + \gamma(\tau^{(q)}) = \max_{1\leq j\leq p} \tau_j + \max_{p+1\leq j\leq p+q} \tau'_j, \\
\theta(\tau^{(p+q)})\gamma(\tau^{(p+q)}) &= \frac{1}{2} \max_{1\leq j\leq p} \tau_j + \max_{p+1\leq j\leq p+q} \tau'_j,
\end{aligned}
$$

or equivalently,

$$
\theta(\tau^{(p+q)}) = \frac{\theta(\tau^{(p)})\gamma(\tau^{(p)}) + \theta(\tau^{(q)})\gamma(\tau^{(q)})}{\gamma(\tau^{(p)}) + \gamma(\tau^{(q)})},
$$

and we conclude that $\mathbf{M}_n^{(p)}$ and $\mathbf{M}_n^{(q)}$ are asymptotically independent.

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