Matrices of local dependence between spatial extreme events

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Abstract: If a spatial process $\{X_{\mathbf{t}}\}_{\mathbf{t}\in\mathbb{Z}^2}$ is isotropic then the usual pairwise extremal dependence measures depend only on the distance $\| \mathbf{i}-\mathbf{j} \|$ between the locations $\mathbf{i}$ and $\mathbf{j}$.

In general, we need to evaluate the spatial dependence in each one of the eight directions of $\mathbb{Z}^2$. We shall consider matrices of multivariate tail and extremal coefficients. We table in matrices the degrees of dependence for chosen pairs of sets $\mathbf{A}$ and $\mathbf{B}$ of locations. The well known relation between the bivariate tail and extremal coefficients can be generalized in this multidirectional approach. The use of such measure-matrices is illustrated in a particular max-stable random field.

1 Introduction

In multivariate and spatial problems attention has often focused on obtaining dependence measures that capture the main characteristics of the dependence structure. For a max--stable stationary random field $\mathbf{X}=\{X_{\mathbf{t}}\}_{\mathbf{t}\in\mathbb{Z}^2}$ the extremal coefficient, $\varepsilon(\mathbf{i},\mathbf{j})$ defined in Schlather [4] and Schlather and Tawn [5] as

$$P\left(\max\left(X_{\mathbf{i}},X_{\mathbf{j}}\right)\leq u\right)=P^{\varepsilon(\mathbf{i},\mathbf{j})}\left(X_{\mathbf{i}}\leq u\right) \tag{1}$$

provides information about pairwise extremal dependence of $\mathbf{X}$.

Unlike a Gaussian process, the dependence structure of a max-stable process is not completely
characterized by pairwise dependence structure. Schlather and Tawn [5] extend the definition of the extremal coefficient to a multivariate setting of any dimension, as follows

$$P\left(\max_{\mathbf{i}\in\mathbf{A}}X_{\mathbf{i}}\leq u\right)=P^{\varepsilon(\mathbf{A})}\left(X_{\mathbf{i}}\leq u\right),\mathbf{A}\subset\mathbb{Z}^2. \tag{2}$$

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This coefficient measures the extremal dependence between the variables indexed by set $\mathbf{A}$. The simple interpretation of $\varepsilon(\mathbf{A})$ as the effective number of independent variables in the set $\mathbf{A}$ from which the maximum is drawn has led to its use as a dependence measure in a range of practical applications.

The extremal coefficient $\varepsilon(\mathbf{A})$ is related to the upper tail dependence parameter defined in Joe ([2]) (see also Li [3] and Ferreira [1]) as

$$\lambda\left(\mathbf{A},\mathbf{B}\right)=\lim_{x\to x^F}P\left(\bigcap_{\mathbf{i}\in\mathbf{A}}\{X_{\mathbf{i}}>x\}\mid\bigcap_{\mathbf{i}\in\mathbf{B}}\{X_{\mathbf{i}}>x\}\right) \tag{3}$$

where, $\mathbf{A}$ and $\mathbf{B}$ are disjoint regions of $\mathbb{Z}^2$, x^F denotes the upper end point of F. This parameter measures the tendency for large values of $\mathbf{X}$ at separate regions of locations to occur simultaneously. If in (3) we take $\mathbf{A}=\{\mathbf{i}\}$ and $\mathbf{B}=\{\mathbf{j}\}$ we obtain the bivariate tail dependence coefficient introduced by Sibuya [6]. In this case, if the probability in (3) is nonzero (zero), *id est*, the most extreme values can (can not) occur together, we say that the random field is pairwise asymptotically dependent (pairwise asymptotically independent) for locations $\mathbf{i}$ and $\mathbf{j}$. The bivariate tail is related with the pairwise extremal coefficient through

$$\lambda\left(\mathbf{i},\mathbf{j}\right)=2-\varepsilon\left(\mathbf{i},\mathbf{j}\right). \tag{4}$$

If the spatial process $\mathbf{X}$ is isotropic then the pairwise extremal dependence measures depend only on the distance $\|\mathbf{i}-\mathbf{j}\|$ between the locations $\mathbf{i}$ and $\mathbf{j}$. In general we don't have isotropy, so we need to evaluate the spatial dependence in each one of the eight directions of $\mathbb{Z}^2$. The aim of this paper is to define matrices of multivariate tail dependence, where each element measures the dependence for particular pair of sets $\mathbf{A}$ and $\mathbf{B}$ of locations. The use of such measure matrices is illustrated in particular max-stable random fields.

2 Characterizing local dependence in extreme events

For each $\mathbf{i}=(i_1,i_2)\in\mathbb{Z}^2$, let $s_j(\mathbf{i}), j=1,2,\ldots,8$, be the neighbors of $\mathbf{i}$ defined as follows:

$$\begin{array}{llll} s_1(\mathbf{i})=(i_1+1,i_2), & s_2(\mathbf{i})=\mathbf{i}+\mathbf{1}, & s_3(\mathbf{i})=(i_1,i_2+1), & s_4(\mathbf{i})=(i_1-1,i_2+1),\\ s_5(\mathbf{i})=(i_1-1,i_2), & s_6(\mathbf{i})=\mathbf{i}-\mathbf{1}, & s_7(\mathbf{i})=(i_1,i_2-1), & s_8(\mathbf{i})=(i_1+1,i_2-1). \end{array}$$

We define the boundary of $A\subseteq\mathbb{Z}^2$ by

$$F_r(\mathbf{A})=\{\mathbf{i}\in\mathbf{A}:s_j(\mathbf{i})\notin\mathbf{A}\quad\text{for some } j\in\{1,2,\ldots 8\}\}$$

and consider

$$T_{s_j}(\mathbf{A})=\{s_j(\mathbf{i}):\mathbf{i}\in F_r(\mathbf{A})\}-\mathbf{A},\quad j=1,2,\ldots,8,$$

as being the set containing the translations of the elements of $Fr(\mathbf{A})$ along the s_j direction.

Our primary measure is a matrix, $\Lambda\left(T_s^k(\mathbf{A}),\mathbf{A}\right)$ of multivariate tail, where its elements measure the tendency for large values of $\mathbf{X}$ at the regions of locations, $\mathbf{A}$ and

$T_{s_j}^k(\mathbf{A}) = \{s_j^k(\mathbf{i}) : \mathbf{i} \in Fr(\mathbf{A})\} - \mathbf{A}, k \geq 2$, where s_j^k denotes $s_j \circ \ldots \circ s_j$, k times, to occur simultaneously.

$$\Lambda\left(T_s^k(\mathbf{A}), \mathbf{A}\right) = \begin{bmatrix} \lambda\left(T_{s_4}^k(\mathbf{A}), \mathbf{A}\right) & \lambda\left(T_{s_3}^k(\mathbf{A}), \mathbf{A}\right) & \lambda\left(T_{s_2}^k(\mathbf{A}), \mathbf{A}\right) \\ \lambda\left(T_{s_5}^k(\mathbf{A}), \mathbf{A}\right) & 1 & \lambda\left(T_{s_1}^k(\mathbf{A}), \mathbf{A}\right) \\ \lambda\left(T_{s_6}^k(\mathbf{A}), \mathbf{A}\right) & \lambda\left(T_{s_7}^k(\mathbf{A}), \mathbf{A}\right) & \lambda\left(T_{s_8}^k(\mathbf{A}), \mathbf{A}\right) \end{bmatrix} \tag{5}$$

where $1 = \lambda(\mathbf{A}, \mathbf{A}) = \lambda(\mathbf{A}, T_{s_0}(\mathbf{A}))$ and $T_{s_0}(\mathbf{A}) = \mathbf{A}$.

In what follows we shall denote the set $\{X_{\mathbf{i}} : \mathbf{i} \in \mathbf{A}\}, \mathbf{A} \subseteq \mathbb{Z}^2$, by $\mathbf{X_A}$ and the vector of dimension $|\mathbf{A}|$ with all components equal x by $\mathbf{x_A}$. The next result presents a recursive formula to obtain the matrices $\Lambda\left(T_s^k(\mathbf{A}), \mathbf{A}\right)$, $k \in \mathbb{N}$.

Proposition 2.1 *If*

$$\alpha\left(T_{s_j}^k(\mathbf{A})\right) = \lim_{\mathbf{x} \to \mathbf{x}^F} \frac{1 - P\left(\mathbf{X}_{T_{s_j}^k(\mathbf{A})} \not> \mathbf{x}_{T_{s_j}^k(\mathbf{A})}\right)}{1 - P\left(\mathbf{X}_{T_{s_j}^k(\mathbf{A})} \leq \mathbf{x}_{T_{s_j}^k(\mathbf{A})}\right)}, k \in \mathbb{N},$$

$$\alpha(\mathbf{A}) = \lim_{\mathbf{x} \to \mathbf{x}^F} \frac{1 - P(\mathbf{X_A} \not> \mathbf{x_A})}{1 - P(\mathbf{X_A} \leq \mathbf{x_A})}, \alpha(\mathbf{A}) > 0$$

and

$$\beta(T_{s_j}^k(\mathbf{A}), \mathbf{A}) = \lim_{\mathbf{x} \to \mathbf{x}^F} \frac{1 - P\left(\mathbf{X}_{T_{s_j}^k(\mathbf{A})} \not> \mathbf{x}_{T_{s_j}^k(\mathbf{A})}, \mathbf{X_A} \not> \mathbf{x_A}\right)}{1 - P\left(\mathbf{X}_{T_{s_j}^k(\mathbf{A})} \leq \mathbf{x_A}, \mathbf{X_A} \leq \mathbf{x_A}\right)}, k \in \mathbb{N}$$

then the coefficients $\lambda(T_{s_j}^k(\mathbf{A}), \mathbf{A}), k \geq 2$, *satisfy*

$$\begin{aligned} \lambda(T_{s_j}^k(\mathbf{A}), \mathbf{A}) \quad = \quad & \lambda(T_{s_j}^{k-1}(\mathbf{A}), \mathbf{A}) - \tfrac{\alpha(T_{s_j}^{k-1}(\mathbf{A}))}{\alpha(\mathbf{A})} \times \tfrac{\varepsilon(T_{s_j}^{k-1}(\mathbf{A}))}{\varepsilon(\mathbf{A})} + \\ & + \tfrac{\beta(T_{s_j}^{k-1}(\mathbf{A}), \mathbf{A})}{\alpha(\mathbf{A})} \times \tfrac{\varepsilon(T_{s_j}^{k-1}(\mathbf{A}) \cup \mathbf{A})}{\varepsilon(\mathbf{A})} + \\ & + \tfrac{\alpha(T_{s_j}^{k}(\mathbf{A}))}{\alpha(\mathbf{A})} \times \tfrac{\varepsilon(T_{s_j}^{k}(\mathbf{A}))}{\varepsilon(\mathbf{A})} - \\ & - \tfrac{\beta(T_{s_j}^{k}(\mathbf{A}), \mathbf{A})}{\alpha(A)} \times \tfrac{\varepsilon(T_{s_j}^{k}(\mathbf{A}) \cup \mathbf{A})}{\varepsilon(\mathbf{A})}. \end{aligned}$$

Proof: *For $k \geq 2$ we have*

$$
\begin{aligned}
\lambda(T_{s_j}^k(\mathbf{A}),\mathbf{A}) &= \lim_{\mathbf{x}\to\mathbf{x}^F}\frac{P\left(\mathbf{X}_{T_{s_j}^k(\mathbf{A})}>\mathbf{x}_\mathbf{A},\mathbf{X}_{T_{s_j}^{k-1}(\mathbf{A})}>\mathbf{x}_{T_{s_j}^{k-1}(\mathbf{A})},\mathbf{X}_\mathbf{A}>\mathbf{x}_\mathbf{A}\right)}{1-P\left(\mathbf{X}_\mathbf{A}\not>\mathbf{x}_\mathbf{A}\right)}+\\
&\quad+\lim_{\mathbf{x}\to\mathbf{x}^F}\frac{P\left(\mathbf{X}_{T_{s_j}^k(\mathbf{A})}>\mathbf{x}_\mathbf{A},\mathbf{X}_{T_{s_j}^{k-1}(\mathbf{A})}\not>\mathbf{x}_{T_{s_j}^{k-1}(\mathbf{A})},\mathbf{X}_\mathbf{A}>\mathbf{x}_\mathbf{A}\right)}{1-P\left(\mathbf{X}_\mathbf{A}\not>\mathbf{x}_\mathbf{A}\right)}=\\
&= \lambda(T_{s_j}^{k-1}(\mathbf{A}),\mathbf{A})-\lim_{\mathbf{x}\to\mathbf{x}^F}\frac{1-P\left(\mathbf{X}_{T_{s_j}^{k-1}(\mathbf{A})}\not>\mathbf{x}_{T_{s_j}^{k-1}(\mathbf{A})}\right)}{1-P\left(\mathbf{X}_\mathbf{A}\not>\mathbf{x}_\mathbf{A}\right)}+\\
&\quad+\lim_{\mathbf{x}\to\mathbf{x}^F}\frac{1-P\left(\mathbf{X}_{T_{s_j}^{k-1}(\mathbf{A})}\not>\mathbf{x}_{T_{s_j}^{k-1}(\mathbf{A})},\mathbf{X}_\mathbf{A}\not>\mathbf{x}_\mathbf{A}\right)}{1-P\left(\mathbf{X}_\mathbf{A}\not>\mathbf{x}_\mathbf{A}\right)}+\\
&\quad+\lim_{\mathbf{x}\to\mathbf{x}^F}\frac{1-P\left(\mathbf{X}_{T_{s_j}^k(\mathbf{A})}\not>\mathbf{x}_{T_{s_j}^k(\mathbf{A})}\right)}{1-P\left(\mathbf{X}_\mathbf{A}\not>\mathbf{x}_\mathbf{A}\right)}-\\
&\quad-\lim_{\mathbf{x}\to\mathbf{x}^F}\frac{1-P(\mathbf{X}_{T_{s_j}^k(\mathbf{A})}\not>\mathbf{x}_{T_{s_j}^k(\mathbf{A})},\mathbf{X}_\mathbf{A}\not>\mathbf{x}_\mathbf{A})}{1-P\left(\mathbf{X}_\mathbf{A}\not>\mathbf{x}_\mathbf{A}\right)}=\\
&= \lambda(T_{s_j}^{k-1}(\mathbf{A}),\mathbf{A})-\frac{\alpha(T_{s_j}^{k-1}(\mathbf{A}))}{\alpha(\mathbf{A})}\times\lim_{\mathbf{x}\to\mathbf{x}^F}\frac{1-P\left(\mathbf{X}_{T_{s_j}^{k-1}(\mathbf{A})}\le\mathbf{x}_{T_{s_j}^{k-1}(\mathbf{A})}\right)}{1-P\left(\mathbf{X}_\mathbf{A}\le\mathbf{x}_\mathbf{A}\right)}+\\
&\quad+\frac{\beta(T_{s_j}^{k-1}(\mathbf{A}),\mathbf{A})}{\alpha(\mathbf{A})}\times\lim_{\mathbf{x}\to\mathbf{x}^F}\frac{1-P\left(\mathbf{X}_{T_{s_j}^{k-1}(\mathbf{A})}\le\mathbf{x}_{T_{s_j}^{k-1}(\mathbf{A})},\mathbf{X}_\mathbf{A}\le\mathbf{x}_\mathbf{A}\right)}{1-P\left(\mathbf{X}_\mathbf{A}\le\mathbf{x}_\mathbf{A}\right)}+\\
&\quad+\frac{\alpha(T_{s_j}^k(\mathbf{A}))}{\alpha(\mathbf{A})}\times\lim_{\mathbf{x}\to\mathbf{x}^F}\frac{1-P\left(\mathbf{X}_{T_{s_j}^k(\mathbf{A})}\le\mathbf{x}_{T_{s_j}^k(\mathbf{A})}\right)}{1-P\left(\mathbf{X}_\mathbf{A}\le\mathbf{x}_\mathbf{A}\right)}-\\
&\quad-\frac{\beta(T_{s_j}^k(\mathbf{A}),\mathbf{A})}{\alpha(\mathbf{A})}\times\lim_{\mathbf{x}\to\mathbf{x}^F}\frac{1-P\left(\mathbf{X}_{T_{s_j}^k(\mathbf{A})}\le\mathbf{x}_{T_{s_j}^k(\mathbf{A})},\mathbf{X}_\mathbf{A}\le\mathbf{x}_\mathbf{A}\right)}{1-P(\mathbf{X}_\mathbf{A}\le\mathbf{x}_\mathbf{A})}.
\end{aligned}
$$

From (2) we obtain the following relation

$$
\begin{aligned}
\lambda(T_{s_j}^k(\mathbf{A}),\mathbf{A}) &= \lambda(T_{s_j}^{k-1}(\mathbf{A}),\mathbf{A})-\frac{\alpha(T_{s_j}^{k-1}(\mathbf{A}))}{\alpha(\mathbf{A})}\times\frac{\varepsilon(T_{s_j}^{k-1}(\mathbf{A}))}{\varepsilon(\mathbf{A})}+\\
&\quad+\frac{\beta(T_{s_j}^{k-1}(\mathbf{A}),\mathbf{A})}{\alpha(\mathbf{A})}\times\frac{\varepsilon(T_{s_j}^{k-1}(\mathbf{A})\cup\mathbf{A})}{\varepsilon(\mathbf{A})}+\\
&\quad+\frac{\alpha(T_{s_j}^k(\mathbf{A}))}{\alpha(\mathbf{A})}\times\frac{\varepsilon(T_{s_j}^k(\mathbf{A}))}{\varepsilon(\mathbf{A})}-\\
&\quad-\frac{\beta(T_{s_j}^k(\mathbf{A}),\mathbf{A})}{\alpha(A)}\times\frac{\varepsilon(T_{s_j}^k(\mathbf{A})\cup\mathbf{A})}{\varepsilon(\mathbf{A})},k\ge 2.
\end{aligned}
$$

The relation (4) can be generalized in our multidirectional approach as

$$\Lambda(T_s^k(\{\mathbf{i}\}),\{\mathbf{i}\})=\mathbf{2}-\varepsilon(\{\mathbf{i}\},T_s^k(\{\mathbf{i}\})),$$

where **2** denote the square matrix with all elements equal 2 and

$$\varepsilon(\{\mathbf{i}\},T_s^k(\{\mathbf{i}\}) = \begin{bmatrix} \varepsilon(\{\mathbf{i}\},T_{s_4}^k(\{\mathbf{i}\}) & \varepsilon(\{\mathbf{i}\},T_{s_3}^k(\{\mathbf{i}\}) & \varepsilon(\{\mathbf{i}\},T_{s_2}^k(\{\mathbf{i}\}) \\ \varepsilon(\{\mathbf{i}\},T_{s_5}^k(\{\mathbf{i}\}) & 1 & \varepsilon(\{\mathbf{i}\},T_{s_1}^k(\{\mathbf{i}\}) \\ \varepsilon(\{\mathbf{i}\},T_{s_6}^k(\{\mathbf{i}\}) & \varepsilon(\{\mathbf{i}\},T_{s_7}^k(\{\mathbf{i}\}) & \varepsilon(\{\mathbf{i}\},T_{s_8}^k(\{\mathbf{i}\}) \end{bmatrix}. \quad (6)$$

3 Examples

We will now illustrate our measure matrices of dependence with a dependent random field. Let $\mathbf{Y} = \{Y_{\mathbf{t}}\}_{\mathbf{t}\in\mathbb{Z}^2}$ be an i.i.d random field with unit Fréchet margins. From $\mathbf{Y}$ we define a stationary and anisotropic random field $\mathbf{X} = \{X_{\mathbf{t}}\}_{\mathbf{t}\in\mathbb{Z}^2}$ as follows

$$X_{\mathbf{i}} = \max\{Y_{s_j(\mathbf{i})}, j \in \{1,\ldots,8\}\}.$$

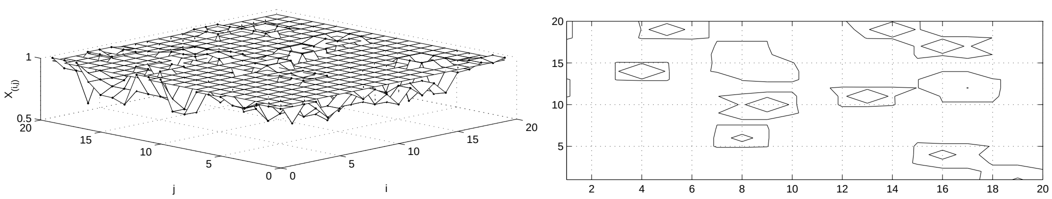

Figure 1: Simulation of the random field $\mathbf{X}$

The matrices of bivariate tail dependence coefficients, until achieved independence in all directions, are

$$\Lambda(T_s^1(\{\mathbf{i}\}),\{\mathbf{i}\}) = \begin{bmatrix} 2/8 & 4/8 & 2/8 \\ 4/8 & 1 & 4/8 \\ 2/8 & 4/8 & 2/8 \end{bmatrix} \quad \Lambda(T_s^2(\{\mathbf{i}\}),\{\mathbf{i}\}) = \begin{bmatrix} 1/8 & 3/8 & 1/8 \\ 3/8 & 1 & 3/8 \\ 1/8 & 3/8 & 1/8 \end{bmatrix} \quad \Lambda(T_s^3(\{\mathbf{i}\}),\{\mathbf{i}\}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

With $k = 3$ we get the independence in all directions. Each one of the last matrices measures the tail dependence between $\{\mathbf{i}\}$ and its neighbors in some step. We illustrate the values of these matrices in figure 2. In that, the light color means less dependence and the dark more dependence, then black is total dependence and white is independence.

The matrices of bivariate extremal coefficients for this random field are

$$\varepsilon(\{\mathbf{i}\},T_s^1(\{\mathbf{i}\})) = \begin{bmatrix} 14/8 & 12/8 & 14/8 \\ 12/8 & 1 & 12/8 \\ 14/8 & 12/8 & 14/8 \end{bmatrix} \quad \varepsilon(\{\mathbf{i}\},T_s^2(\{\mathbf{i}\})) = \begin{bmatrix} 15/8 & 13/8 & 15/8 \\ 13/8 & 1 & 13/8 \\ 15/8 & 13/8 & 15/8 \end{bmatrix} \quad \varepsilon(\{\mathbf{i}\},T_s^3(\{\mathbf{i}\})) = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

The measure of the tendency for large values of $\mathbf{X}$ at the regions $\mathbf{A} = \{\mathbf{i}, \mathbf{i}+\mathbf{1}\}$ ant its translation along each direction, to occur simultaneously, is given by

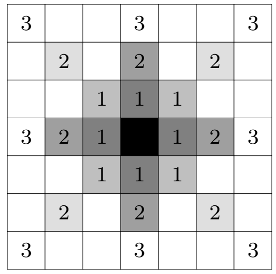

Figure 2: Representation of matrices values of bivariate tail dependence coefficients for $\mathbf{Y}$ at steps 1, 2 and 3.

$$\Lambda(T_s^1(\mathbf{A}), \mathbf{A}) = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix},$$

and the independence in all directions was achieved in two steps,

$$\Lambda(T_s^2(\mathbf{A}), \mathbf{A}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

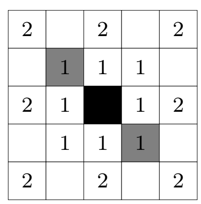

Figure 3: Representation of matrices values of the tail dependence coefficients for the sets of locations $\mathbf{A} = \{\mathbf{i}, \mathbf{i+1}\}$ and $T_s^k(\mathbf{A})$, $k = 1, 2$.

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