Measuring dependence of two multivariate extremes

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Abstract: We extend the characterizations given by Takahashi for the independence and the total dependence of the univariate marginals of a multivariate extreme value distribution to its multivariate marginals. We also deal with the problem of how to measure the strength of the dependence among multivariate extremes. By presenting new definitions for extremal and tail coefficients, we propose measures that summarize the dependence between two multivariate extreme value distributions and preserve the main properties of the known bivariate coefficients for two univariate extreme value distributions. Finally, we illustrate these contributions to model the dependence among multivariate marginals with examples.

Keywords: independence, total dependence, multivariate extreme value distribution, multivariate marginals, dependence coefficients

1 Introduction

Consider a random vector $\mathbf{X} = (X_1, \ldots, X_{p+q})$ with multivariate extreme value distribution $G_{\mathbf{X}}$, G_{X_j} being its j-th univariate marginal distribution.

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For each integer $d > 1$, let $(x_1, \ldots, x_d) \in \mathbb{R}^d$ be denoted by $\mathbf{x}^{(d)}$ and, for $\mathbf{a}^{(d)}, \mathbf{b}^{(d)} \in \mathbb{R}^d$ and $c \in \mathbb{R}$, $\mathbf{a}^{(d)} + c\mathbf{b}^{(d)} = (a_1 + cb_1, \ldots, a_d + cb_d)$, $\mathbf{a}^{(d)^c} = (a_1^c, \ldots, a_d^c)$, $\mathbf{a}^{(d)} \leq \mathbf{b}^{(d)}$ if and only if $a_j \leq b_j$ for all $j = 1, \ldots, d$, and $\mathbf{a}^{(d)} \not\leq \mathbf{b}^{(d)}$ if and only if $a_j > b_j$ for some $j = 1, \ldots, d$.

Takahashi (1988) proved that $G_{\mathbf{X}}(\mathbf{x}^{(p+q)}) = \prod_{j=1}^{p+q} G_{X_j}(x_j)$, for each $\mathbf{x}^{(p+q)} \in \mathbb{R}^{p+q}$, if and only if there exists $\mathbf{y}^{(p+q)} \in \mathbb{R}^{p+q}$ such that $G_{\mathbf{X}}(\mathbf{y}^{(p+q)}) = \prod_{j=1}^{p+q} G_{X_j}(y_j) \in (0,1)$.

The condition for the total positive dependence is $G_{\mathbf{X}}(\mathbf{x}^{(p+q)}) = \min_{1 \leq j \leq p+q} G_{X_j}(x_j)$ if and only if there exists $\mathbf{y}^{(p+q)} \in \mathbb{R}^{p+q}$ such that $G_{\mathbf{X}}(\mathbf{y}^{(p+q)}) = G_{X_1}(y_1) = \ldots = G_{X_{p+q}}(y_{p+q}) \in (0,1)$.

Let $\mathbf{X}^{(p)} = (X_1, \ldots, X_p)$ and $\mathbf{X}^{(q)} = (X_{p+1}, \ldots, X_{p+q})$ be two sub-vectors of $\mathbf{X}$. We show that the above simple conditions can be extended to characterize the independence and the total dependence of $\mathbf{X}^{(p)}$ and $\mathbf{X}^{(q)}$. We state the results for two vectors for sake of simplicity. They can be rewritten for several vectors including the case of $p+q$ variables corresponding to the result of Takahashi. However the arguments in the proof of the characterization of the independence or the total dependence of univariate marginals do not apply to the analogous characterization for vectors in a straightforward way.

Both parametric and nonparametric modelling of dependence has been well established and probabilistic as well statistical aspects of the problem laid out. However, in order to quantify such dependence, the extremal coefficient and the tail dependence coefficients defined for bivariate or multivariate distributions have been considered to model dependence among univariate marginals (see Joe (1997), Schmidt (2002), Frahm (2006), Schmid and Schmidt (2007) and refeences therein).

We will consider dependence between multivariate marginals and present a tail dependence coefficient which gives the probability that the realization of several random variables are extremely large under the condition that other variables are also extremely positive.

An analogous lower tail dependence coefficient can be defined for extremely negative realizations. These measures summarize the asymptotic dependence of two multivariate extreme distributions and have the useful tail dependence coefficient λ (Joe(1993)) as a special case.

We also extend the extremal coefficient ϵ (Sibuya (1960)) in two directions and explore the relation between λ and ϵ. Both coefficients, presented in Sections 3 and 4, give information about the contribution of a part of the random vector in the strength of dependence of the multivariate extreme value (MEV) distribution. The results of section 4 can also be stated for two or more vectors.

In section 5 we compute some coefficients in two simple examples. A directory of such coefficients for distributions with practical interest will be presented in a forthcoming work.

2 Characterization of independence and total dependence

Throughout this paper $G_{\mathbf{Y}}$ and $D_{\mathbf{Y}}$ denote the distribution function and the dependence function (or copula) of a random vector $\mathbf{Y} = (Y_1, \ldots, Y_d)$, respectively. We shall consider vectors $\mathbf{Y}$ with MEV distribution and we henceforth call dependence function of $\mathbf{Y}$ the informative function $D_{\mathbf{Y}}(u_1, \ldots, u_d) = P(G_{Y_1}(Y_1) \leq u_1, \ldots, G_{Y_d}(Y_d) \leq u_d)$, $\mathbf{u}^{(d)} \in [0,1]^d$. Complete characterizations of the extreme dependence functions are given by Sibuya (1960), Deheuvels(1978) and Hsing (1989), among others.

The following propositions present a characterization for the independence and the total dependence of the multivariate marginals $\mathbf{X}^{(p)}$ and $\mathbf{X}^{(q)}$ of $\mathbf{X}$ with MEV distribution.

Proposition 2.1 *For* $\mathbf{X} = (X_1, \ldots, X_{p+q})$ *with MEV distribution,* $\mathbf{X}^{(p)} = (X_1, \ldots, X_p)$ *and* $\mathbf{X}^{(q)} = (X_{p+1}, \ldots, X_{p+q})$ *it holds* $G_{\mathbf{X}}(\mathbf{x}^{(p+q)}) = G_{\mathbf{X}^{(p)}}(\mathbf{x}^{(p)})G_{\mathbf{X}^{(q)}}(\mathbf{x}^{(q)})$ *for all* $\mathbf{x}^{(p+q)} \in \mathbb{R}^{p+q}$ *if and only if there exists* $\mathbf{y}^{(p+q)} \in \mathbb{R}^{p+q}$ *such that* $0 < G_{\mathbf{X}^{(p)}}(\mathbf{y}^{(p)}) < 1$, $0 < G_{\mathbf{X}^{(q)}}(\mathbf{y}^{(q)}) < 1$ *and* $G_{\mathbf{X}}(\mathbf{y}^{(p+q)}) = G_{\mathbf{X}^{(p)}}(\mathbf{y}^{(p)})G_{\mathbf{X}^{(q)}}(\mathbf{y}^{(q)})$.

Proof: Suppose that $x_i \geq y_i$, $i = 1, \ldots, p+q$. Let $\{\mathbf{a}_n^{(p+q)} > \mathbf{0}\}_{n\geq 1}$, $\{\mathbf{b}_n^{(p+q)}\}_{n\geq 1}$ and $\{\mathbf{X}_n^{(p+q)}\}_{n\geq 1}$ be such that

$$P^n(\mathbf{X}_n^{(p+q)} \leq \mathbf{a}_n^{(p+q)}\mathbf{x}^{(p+q)} + \mathbf{b}_n^{(p+q)}) \underset{n\to\infty}{\longrightarrow} G_{\mathbf{X}}(\mathbf{x}^{(p+q)}), \quad \mathbf{x}^{(p+q)} \in \mathbb{R}^{p+q}. \tag{2.1}$$

Since

$$n\left(1 - P(\mathbf{X}_1^{(p)} \leq \mathbf{a}_n^{(p)}\mathbf{y}^{(p)} + \mathbf{b}_n^{(p)}, \mathbf{X}_1^{(q)} \leq \mathbf{a}_n^{(q)}\mathbf{y}^{(q)} + \mathbf{b}_n^{(q)})\right) \underset{n\to\infty}{\longrightarrow} -\log G_{\mathbf{X}^{(p)}}(\mathbf{y}^{(p)}) - \log G_{\mathbf{X}^{(q)}}(\mathbf{y}^{(q)}), \tag{2.2}$$

we have

$$nP(\mathbf{X}_1^{(p)} \not\leq \mathbf{a}_n^{(p)}\mathbf{x}^{(p)} + \mathbf{b}_n^{(p)}, \mathbf{X}_1^{(q)} \not\leq \mathbf{a}_n^{(q)}\mathbf{x}^{(q)} + \mathbf{b}_n^{(q)}) \leq$$
$$nP(\mathbf{X}_1^{(p)} \not\leq \mathbf{a}_n^{(p)}\mathbf{y}^{(p)} + \mathbf{b}_n^{(p)}, \mathbf{X}_1^{(q)} \not\leq \mathbf{a}_n^{(q)}\mathbf{y}^{(q)} + \mathbf{b}_n^{(q)}) \underset{n\to\infty}{\longrightarrow} 0$$

and then (2.2) holds for $\mathbf{x}^{(p+q)}$. This implies to (2.1) with $G_{\mathbf{X}}(\mathbf{x}^{(p+q)}) = G_{\mathbf{X}^{(p)}}(\mathbf{x}^{(p)})G_{\mathbf{X}^{(q)}}(\mathbf{x}^{(q)})$. Therefore we have proved that $D_{\mathbf{X}}(\mathbf{u}^{(p+q)}) = D_{\mathbf{X}^{(p)}}(\mathbf{u}^{(p)})D_{\mathbf{X}^{(q)}}(\mathbf{u}^{(q)})$, for all $\mathbf{u}^{(p+q)}$ such that $u_j \geq G_{X_j}(y_j)$, $j = 1, \ldots, p+q$.

If $u_j < G_{X_j}(y_j)$ for some j then, from the max-stability of each G_{X_j}, there exists $s > 0$ such that $u_j^{1/s} \geq G_{X_j}(y_j)$, $j = 1, \ldots, p+q$, and we also have

$$D_{\mathbf{X}}(\mathbf{u}^{(p+q)}) = D^s_{\mathbf{X}}(\mathbf{u}^{(p+q)^{1/s}}) = D^s_{\mathbf{X}^{(p)}}(\mathbf{u}^{(p)^{1/s}})D^s_{\mathbf{X}^{(q)}}(\mathbf{u}^{(q)^{1/s}}) = D_{\mathbf{X}^{(p)}}(\mathbf{u}^{(p)})D_{\mathbf{X}^{(q)}}(\mathbf{u}^{(q)}).$$

□

Proposition 2.2 *Let $\mathbf{X} = (X_1, \ldots, X_{p+q})$ have a MEV distribution.*

i) *If there exists $\mathbf{y}^{(p+q)} \in \mathbb{R}^{p+q}$ such that $G_{\mathbf{X}}(\mathbf{y}^{(p+q)}) = G_{X_1}(y_1) = \ldots = G_{X_{p+q}}(y_{p+q}) = a \in (0,1)$ then, for each two sub-vectors $\mathbf{X}^{(s)}$ and $\mathbf{X}^{(t)}$ of $\mathbf{X}$, with $s + t = p + q$, it holds $G_{\mathbf{X}}(\mathbf{x}^{(p+q)}) = \min\{G_{\mathbf{X}^{(s)}}(\mathbf{x}^{(s)}), G_{\mathbf{X}^{(t)}}(\mathbf{x}^{(t)})\}$ for all $\mathbf{x}^{(p+q)} \in \mathbb{R}^{p+q}$.*

ii) *If $G_{\mathbf{X}}(\mathbf{x}^{(p+q)}) = \min\{G_{\mathbf{X}^{(p)}}(\mathbf{x}^{(p)}), G_{\mathbf{X}^{(q)}}(\mathbf{x}^{(q)})\}$ for all $\mathbf{x}^{(p+q)} \in \mathbb{R}^{p+q}$ then there exists $\mathbf{y}^{(p+q)} \in \mathbb{R}^{p+q}$ such that $G_{\mathbf{X}}(\mathbf{y}^{(p+q)}) = G_{\mathbf{X}^{(p)}}(\mathbf{y}^{(p)}) = G_{\mathbf{X}^{(q)}}(\mathbf{y}^{(q)}) = G_{X_1}(y_1) = \ldots = G_{X_{p+q}}(y_{p+q}) = a \in (0,1)$.*

Proof: (i) The result can be obtained as a corollary of the Takahashy's characterization for total positive dependence of the marginals of $\mathbf{X}$. However we present a direct proof. Let $z = \min\{G_{\mathbf{X}^{(p)}}(\mathbf{x}^{(p)}), G_{\mathbf{X}^{(q)}}(\mathbf{x}^{(q)})\}$ and $s > 0$ such that $a^s = z$. We have, for each $j = 1, \ldots, p+q$,

$$G_{X_j}(x_j) \geq z = G^s_{X_j}(y_j)$$

and then

$$z \geq G_{\mathbf{X}}(\mathbf{x}^{(p+q)}) = D_{\mathbf{X}}(G_{X_1}(x_1), \ldots, G_{X_{p+q}}(x_{p+q})) \geq$$

$$D_{\mathbf{X}}(G^s_{X_1}(y_1), \ldots, G^s_{X_{p+q}}(y_{p+q})) =$$

$$D^s_{\mathbf{X}}(G_{X_1}(y_1), \ldots, G_{X_{p+q}}(y_{p+q})) = a^s = z.$$

(ii) If $G_{\mathbf{X}}(\mathbf{x}^{(p+q)}) = \min\{G_{\mathbf{X}^{(p)}}(\mathbf{x}^{(p)}), G_{\mathbf{X}^{(q)}}(\mathbf{x}^{(q)})\}$ for all $\mathbf{x}^{(p+q)} \in \mathbb{R}^{p+q}$ then, by applying the Theorem 3.5.3 in Nelsen (2006), it holds $G_{\mathbf{X}^{(p)}}(\mathbf{x}^{(p)}) = \min\{G_{X_j}(x_j), j = 1, \ldots, p\}$ for all $\mathbf{x}^{(p)} \in \mathbb{R}^p$, $G_{\mathbf{X}^{(q)}}(\mathbf{x}^{(q)}) = \min\{G_{X_j}(x_j), j = p+1, \ldots, q\}$ for all $\mathbf{x}^{(q)} \in \mathbb{R}^q$ and the marginals of $\mathbf{X}$ are totally dependent. Therefore, by applying again the Takahashy's result there exists $\mathbf{y}^{(p+q)} \in \mathbb{R}^{p+q}$ such that $G_{\mathbf{X}}(\mathbf{y}^{(p+q)}) = G_{X_1}(y_1) = \ldots = G_{X_{p+q}}(y_{p+q}) = a \in (0,1)$. For such $\mathbf{y}^{(p+q)}$ we also have $G_{\mathbf{X}^{(p)}}(\mathbf{y}^{(p)}) = G_{\mathbf{X}^{(q)}}(\mathbf{y}^{(q)}) = a$. □

The example presented at the end of the Section 5 shows that the existence of a point $\mathbf{y}^{(p+q)} \in \mathbb{R}^{p+q}$ such that $G_{\mathbf{X}}(\mathbf{y}^{(p+q)}) = G_{\mathbf{X}^{(p)}}(\mathbf{y}^{(p)}) = G_{\mathbf{X}^{(q)}}(\mathbf{y}^{(q)}) = a$ is not sufficient to guarantee that $\mathbf{x}^{(p)}$ and $\mathbf{x}^{(q)}$ are totally dependent.

3 Tail dependence

For $\mathbf{X} = (X_1, X_2)$ and $G_{X_1} = G_{X_2} = F$, the tail dependence coefficient (Sibuya (1960), Joe (1993)) is defined by

$$\lambda = \lim_{x \to \omega(F)} P(X_2 > x | X_1 > x) \tag{3.3}$$

where $\omega(F)$ denotes the upper end point of F.

It holds

$$\lambda = 2 - \lim_{u \to 1} \frac{\log D_{G_{\mathbf{X}}}(u, u)}{\log u} \tag{3.4}$$

where $\epsilon = \lim_{u \to 1} \frac{\log D_{G_{\mathbf{X}}}(u, u)}{\log u}$ is called the extremal coefficient. We will return to the extremal coefficient in the next sections.

Hereinafter let $F = G_{X_j}$, $j = 1, \ldots, p+q$ and $\mathbf{x}^{(p+q)} = (x, \ldots, x)$, $x \in \mathbb{R}$.

The following definition for tail dependence coefficient generalizes (3.3) and its interpretation.

Definition 3.1 *Let* $\mathbf{X} = (X_1, \ldots, X_{p+q})$ *have a MEV distribution with equal marginal distributions and* $\mathbf{X}^{(p)}$ *and* $\mathbf{X}^{(q)}$ *be sub-vectors of* $\mathbf{X}$. *The coefficient* $\lambda_{\mathbf{X}^{(p)}}^{\mathbf{X}^{(q)}}$ *is defined as*

$$\lambda_{\mathbf{X}^{(p)}}^{\mathbf{X}^{(q)}} = \lim_{x \to \omega(F)} P(\mathbf{X}^{(q)} > \mathbf{x}^{(q)} | \mathbf{X}^{(p)} > \mathbf{x}^{(p)}),$$

provided the limit exists.

If, for some vectors $\mathbf{X}^{(s)}$ and $\mathbf{X}^{(t)}$, with $s + t = p + q$, the above limit is positive then the random vector $\mathbf{X}$ is said multivariate tail dependent accordingly the definition 7.1 in Schmidt (2002). If this limit is zero for all subvectors this author considers that the random vector is multivariate tail independent.

Proposition 3.1 *If* $\alpha^{(s)} = \lim_{x \to \omega(F)} \frac{1 - P(\mathbf{X}^{(s)} \not> \mathbf{x}^{(s)})}{1 - P(\mathbf{X}^{(s)} \leq \mathbf{x}^{(s)})}$, $s = p, q$, $\alpha^{(p)} > 0$ *and* $\beta^{(p,q)} = \lim_{x \to \omega(F)} \frac{1 - P(\mathbf{X}^{(p)} \not> \mathbf{x}^{(p)}, \mathbf{X}^{(q)} \not> \mathbf{x}^{(q)})}{1 - P(\mathbf{X} \leq \mathbf{x}^{(p+q)})}$, *then the coefficient* $\lambda_{\mathbf{X}^{(p)}}^{\mathbf{X}^{(q)}}$ *satisfies*

$$\lambda_{\mathbf{X}^{(p)}}^{\mathbf{X}^{(q)}} = 1 + \tfrac{\alpha^{(q)}}{\alpha^{(p)}} \lim_{u \to 1} \frac{\log D_{\mathbf{X}^{(q)}}(\mathbf{u}^{(q)})}{\log D_{\mathbf{X}^{(p)}}(\mathbf{u}^{(p)})} - \frac{\beta^{(p,q)}}{\alpha^{(p)}} \lim_{u \to 1} \frac{\log D_{\mathbf{X}}(\mathbf{u}^{(p+q)})}{\log D_{\mathbf{X}^{(p)}}(\mathbf{u}^{(p)})}.$$

Proof: Since, for each j, $\lim_{x \to \omega(F)} P(\mathbf{X}^{(p+q)} \not> \mathbf{x}^{(p+q)}, X_j > x) = 0$ we have $\lim_{x \to \omega(F)} P(\mathbf{X}^{(p+q)} \not> \mathbf{x}^{(p+q)}) = \lim_{x \to \omega(F)} P(\mathbf{X}^{(p+q)} \leq \mathbf{x}^{(p+q)})$ and then

$$\lambda_{\mathbf{X}^{(p)}}^{\mathbf{X}^{(q)}} = 1 + \frac{\alpha^{(q)}}{\alpha^{(p)}} \lim_{x \to \omega(F)} \frac{1 - P(\mathbf{X}^{(q)} \leq \mathbf{x}^{(q)})}{1 - P(\mathbf{X}^{(p)} \leq \mathbf{x}^{(p)})} - \frac{\beta^{(p,q)}}{\alpha^{(p)}} \lim_{x \to \omega(F)} \frac{1 - P(\mathbf{X}^{(p+q)} \leq \mathbf{x}^{(p+q)})}{1 - P(\mathbf{X}^{(p)} \leq \mathbf{x}^{(p)})} =$$

$$1 + \frac{\alpha^{(q)}}{\alpha^{(p)}} \lim_{u \to 1} \frac{\log D_{\mathbf{X}^{(q)}}(\mathbf{u}^{(q)})}{\log D_{\mathbf{X}^{(p)}}(\mathbf{u}^{(p)})} - \frac{\beta^{(p,q)}}{\alpha^{(p)}} \lim_{u \to 1} \frac{\log D_{\mathbf{X}}(\mathbf{u}^{(p+q)})}{\log D_{\mathbf{X}^{(p)}}(\mathbf{u}^{(p)})}.$$

□

The rates $\alpha^{(s)}$, $s = p, q$ and $\beta^{(p,q)}$ are not in general equal to one, as is illustrated in the last example of Section 6. When $p = q = 1$ we find in the above proposition the relation (3.4).

4 The coefficient $\epsilon_{\mathbf{X}^{(p)}}^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})}$

The extremal coefficient considered by Smith (1990) is defined as the constant $\epsilon \in [1, d]$ such that

$$G_{\mathbf{X}}(x, \ldots, x) = F^{\epsilon}(x), \; x \in \mathbb{R}. \tag{4.5}$$

where F denotes the common extreme distribution marginal of $G_{\mathbf{X}}$.

This expressive indicator of extremal dependence generalizes the one considered by Tiago de Oliveira (1962/63) for bivariate extreme distributions and is a particular case of the function considered in Buishand (1984). It takes the extreme values 1 or d if and only if $\mathbf{X}$ has totally dependent or independent marginals, respectively.

For $d = 2$ the extremal coefficient can be related with the tail dependence parameter λ as mentioned before.

We shall denote this coefficient for a vector $\mathbf{X}$ by $\epsilon^{\mathbf{X}}$. Since

$$P(\mathbf{X}^{(p)} \leq \mathbf{x}^{(p)}, \mathbf{X}^{(q)} \leq \mathbf{x}^{(q)}) = F^{\epsilon^{\mathbf{X}}}(x) = D^{\alpha}_{\mathbf{X}^{(p)}}(F(x), \ldots, F(x))$$

with $\alpha = \frac{\epsilon^{\mathbf{X}}}{\epsilon^{\mathbf{X}^{(p)}}}$, we shall consider this parameter to measure the contribution of $\mathbf{X}^{(p)}$ in the dependence structure of $\mathbf{X}$.

Definition 4.1 *Let $\mathbf{X} = (X_1, \ldots, X_{p+q})$ have a MEV distribution with equal marginal distributions and $\mathbf{X}^{(p)}$ and $\mathbf{X}^{(q)}$ be sub-vectors of $\mathbf{X}$. The coefficient $\epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})}_{\mathbf{X}^{(p)}}$ is defined as*

$$\epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})}_{\mathbf{X}^{(p)}} = \frac{\epsilon^{\mathbf{X}}}{\epsilon^{\mathbf{X}^{(p)}}}.$$

When $p = q = 1$ we find in the above definition the extremal coefficient for bivariate distributions and its properties are also stated in the next proposition.

Proposition 4.1

i) $\epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})}_{\mathbf{X}^{(p)}} = \lim_{u \to 1} \dfrac{\log D_{\mathbf{X}}(\mathbf{u}^{(p+q)})}{\log D_{\mathbf{X}^{(p)}}(\mathbf{u}^{(p)})}$.

ii) $1 \leq \epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})}_{\mathbf{X}^{(p)}} \leq 1 + \frac{\epsilon^{\mathbf{X}^{(q)}}}{\epsilon^{\mathbf{X}^{(p)}}} \leq 1 + q$.

iii) $\lambda^{\mathbf{X}^{(q)}}_{\mathbf{X}^{(p)}} = 1 + \frac{\alpha^{(q)}}{\alpha^{(p)}} \frac{\epsilon^{\mathbf{X}^{(q)}}}{\epsilon^{\mathbf{X}^{(p)}}} - \frac{\beta^{(p,q)}}{\alpha^{(p)}} \epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})}_{\mathbf{X}^{(p)}}$.

iv) *If $p = q$, $G_{\mathbf{X}^{(p)}} = G_{\mathbf{X}^{(q)}}$ and $\alpha^{(p)} > 0$ then $\epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})}_{\mathbf{X}^{(p)}} = \epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})}_{\mathbf{X}^{(q)}}$, $\lambda^{\mathbf{X}^{(q)}}_{\mathbf{X}^{(p)}} = \lambda^{\mathbf{X}^{(p)}}_{\mathbf{X}^{(q)}}$ and*

$$\lambda^{\mathbf{X}^{(q)}}_{\mathbf{X}^{(p)}} = 2 - \frac{\beta^{(p,q)}}{\alpha^{(p)}} \epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})}_{\mathbf{X}^{(p)}} = 2 - \frac{\beta^{(p,q)}}{\alpha^{(q)}} \epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})}_{\mathbf{X}^{(q)}}.$$

Proof: (i) follows from the definition of the extremal coefficient ϵ.

(ii) Since $\epsilon^{\mathbf{X}^{(p)}} \leq \epsilon^{\mathbf{X}}$ for each sub-vector $\mathbf{X}^{(p)}$ of $\mathbf{X}$, we have the first inequality. The association of $\mathbf{X}^{(p)}$ and $\mathbf{X}^{(q)}$ implies $\epsilon^{\mathbf{X}} \leq \epsilon^{\mathbf{X}^{(p)}} + \epsilon^{\mathbf{X}^{(q)}}$ which leads to the second inequality.

(iii) and (iv) derive from the Proposition 3.1. □

The upper bound $1+q$ in (ii) can be attained. Consider for instance $\mathbf{X} = (X_1, X_2, X_3, X_4)$, $\mathbf{X}^{(p)} = (X_1, X_2)$ and $\mathbf{X}^{(q)} = (X_3, X_4)$ with $X_1 = X_2 = Y$, $X_3 = Z$ and $X_4 = W$ independent and with the same extreme value distribution. Then $\epsilon^{\mathbf{X}^{(p)}} = 1$, $\epsilon^{\mathbf{X}^{(q)}} = 2$ and $\epsilon^{\mathbf{X}} = 3$. The lower bound is attained if for example $\mathbf{X}$ has totally dependent marginals.

5 The coefficient $\epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})}$

Since $P(\mathbf{X}^{(p)} \leq \mathbf{x}^{(p)}, \mathbf{X}^{(q)} \leq \mathbf{x}^{(q)}) = (G_{\mathbf{X}^{(p)}}(\mathbf{x}^{(p)}) G_{\mathbf{X}^{(q)}}(\mathbf{x}^{(q)}))^{\beta}$ with $\beta = \frac{\epsilon^{\mathbf{X}}}{\epsilon^{\mathbf{X}^{(p)}} + \epsilon^{\mathbf{X}^{(q)}}}$ we propose to measure the dependence between $\mathbf{X}^{(p)}$ and $\mathbf{X}^{(q)}$ by using this coefficient.

Definition 5.1 *Let $\mathbf{X} = (X_1, \ldots, X_{p+q})$ have a MEV distribution with equal marginal distributions and $\mathbf{X}^{(p)}$ and $\mathbf{X}^{(q)}$ be sub-vectors of $\mathbf{X}$. The coefficient $\epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})}$ is defined as*

$$\epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})} = \frac{\epsilon^{\mathbf{X}}}{\epsilon^{\mathbf{X}^{(p)}} + \epsilon^{\mathbf{X}^{(q)}}}.$$

When $p = q$ and $G_{\mathbf{X}^{(p)}} = G_{\mathbf{X}^{(q)}}$ it holds $\epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})} = \frac{1}{2}\epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})}_{\mathbf{X}^{(p)}}$.

We emphasize that, when $p = q = 1$, it holds $G_{\mathbf{X}}(x, x) = (F(x)F(x))^{\epsilon^{\mathbf{X}}/2}$ and we shall call $\epsilon^{\mathbf{X}}/2 \in [\frac{1}{2}, 1]$ the rescaled extremal coefficient. Therefore the definition proposed here is a generalization of the rescaled extremal coefficient and the main properties of this coefficient are stated in the following proposition. Both coefficients, the Smith's extremal coefficient ϵ and its generalization here proposed, are decreasing with respect to the strength of dependence.

Proposition 5.1

i) $\epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})} = \lim_{u\to 1} \dfrac{\log D_{\mathbf{X}}(\mathbf{u}^{(p+q)})}{\log D_{\mathbf{X}^{(p)}}(\mathbf{u}^{(p)}) + \log D_{\mathbf{X}^{(q)}}(\mathbf{u}^{(q)})}$.

ii) $\epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})} \in [\frac{1}{2}, 1]$.

iii) $\epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})} = 1$ *if and only if* $\mathbf{X}^{(p)}$ *and* $\mathbf{X}^{(q)}$ *are independent.*

iv) *If* $\mathbf{X}^{(p)}$ *and* $\mathbf{X}^{(q)}$ *are totally dependent then* $\epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})} = \frac{1}{2}$.

v) $\left(\epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})}\right)^{-1} = \left(\epsilon_{\mathbf{X}^{(p)}}^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})}\right)^{-1} + \left(\epsilon_{\mathbf{X}^{(q)}}^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})}\right)^{-1}$.

vi) $\lambda_{\mathbf{X}^{(p)}}^{\mathbf{X}^{(q)}} = 1 + \frac{\alpha^{(q)}}{\alpha^{(p)}}\frac{\epsilon^{\mathbf{X}^{(q)}}}{\epsilon^{\mathbf{X}^{(p)}}} - \frac{\beta^{(p,q)}}{\alpha^{(p)}}\frac{\epsilon^{\mathbf{X}^{(p)}}+\epsilon^{\mathbf{X}^{(q)}}}{\epsilon^{\mathbf{X}^{(p)}}}\epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})}$ *and, in particular when* $G_{\mathbf{X}^{(p)}} = G_{\mathbf{X}^{(q)}}$, $\lambda_{\mathbf{X}^{(p)}}^{\mathbf{X}^{(q)}} = 2 - 2\frac{\beta^{(p,q)}}{\alpha^{(p)}}\epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})}$.

The proof of (iii) and (iv) applies the definition of the coefficients and the propositions 2.1 and 2.2. To obtain (ii) we apply the arguments used in (ii) of the previous proposition.

We remark that if $\epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})} = \frac{1}{2}$ then we can not guarantee that $\mathbf{X}^{(p)}$ and $\mathbf{X}^{(q)}$ are totally dependent as show the following example. Let $\mathbf{X} = (X_1, X_2, X_3, X_4, X_5, X_6)$, $\mathbf{X}^{(p)} = (X_1, X_2, X_3)$ and $\mathbf{X}^{(q)} = (X_4, X_5, X_6)$ where $X_1 = X_4 = X_5 = Y$ and $X_2 = X_3 = X_6 = Z$ with Y and Z Gumbel independent variables. Then $\epsilon^{\mathbf{X}^{(p)}} = \epsilon^{\mathbf{X}^{(q)}} = \epsilon^{\mathbf{X}} = 2$ and we can easly find $\mathbf{x}^{(6)} \in \mathbb{R}^6$ (for example $x_1 = x_2 = x_3 = x_4 = x_5 = 0 > x_6$) for which $G_{\mathbf{X}}(\mathbf{x}^{(6)}) \neq \min\{G_{\mathbf{X}^{(p)}}(\mathbf{x}^{(p)}), G_{\mathbf{X}^{(q)}}(\mathbf{x}^{(q)})\}$. We can also present $\mathbf{y}^{(6)}$ (for example with all the components equal to zero) such that $G_{\mathbf{X}}(\mathbf{y}^{(6)}) = G_{\mathbf{X}^{(p)}}(\mathbf{y}^{(p)}) = G_{\mathbf{X}^{(q)}}(\mathbf{y}^{(q)}) = a \in$

$(0,1)$. Therefore the existance of such kind of point is not sufficient to guarantee that $\mathbf{X}^{(p)}$ and $\mathbf{X}^{(q)}$ are totally dependent.

6 Examples

The two examples we will consider are particular cases of the dependence function of Family MM3 in Joe (1997).

Example 6.1. Let the MEV dependence function $D_{\mathbf{X}}$ defined by

$$D_{\mathbf{X}}(\mathbf{u}^{(4)}) = exp\{-\sum_{j=1}^{4} z_j + \sum_{i=1}^{4}\sum_{j=i+1}^{4} ((p_i z_i)^{-\delta_{ij}} + (p_j z_j)^{-\delta_{ij}})^{-\frac{1}{\delta_{ij}}}\}$$

where $z_j = -\log u_j$, $p_j \geq 0$ and $\delta_{ij} > 0$ are constants, $i, j = 1, \ldots, 4$.

The bivariate tail dependence coefficient λ for (X_i, X_j) is

$$\lambda = (p_i^{-\delta_{ij}} + p_j^{-\delta_{ij}})^{-1/\delta_{ij}}, \quad i, j = 1, \ldots, 4. \tag{6.6}$$

Let consider the copula obtained by taking $2p_2 = p_1 = p_3 = p_4 = 1$, $\delta_{1,2} = \delta_{3,4} = 1$ and $\delta_{1,3} = \delta_{1,4} = \delta_{2,3} = \delta_{2,4} = \delta \to 0$. Then

$$D_{\mathbf{X}}(\mathbf{u}^{(4)}) = exp\{4\log u + (3(-\log u)^{-1})^{-1} + (2(-\log u)^{-1})^{-1}\}$$

and, for $\mathbf{X}^{(p)} = (X_1, X_2)$ and $\mathbf{X}^{(q)} = (X_3, X_4)$, we have

$$\epsilon^{\mathbf{X}^{(p)}} = \lim_{u \to 1} \frac{2\log u + \frac{1}{3}\log\frac{1}{u}}{\log u} = \frac{5}{3},$$

$$\epsilon^{\mathbf{X}^{(q)}} = \frac{3}{2}, \quad \epsilon^{\mathbf{X}} = \frac{19}{6}, \quad \epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})}_{\mathbf{X}^{(p)}} = \frac{19}{10}.$$

Moreover we obtain $\epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})} = 1$ and then $\lambda^{\mathbf{X}^{(p)}}_{\mathbf{X}^{(q)}} = 0$, as expected from the choice of δ_{ij} and (6.6).

If we take for instance $p_4 = 2$ then $\mathbf{X}^{(p)}$ and $\mathbf{X}^{(q)}$ have the same distribution and $\epsilon_{\mathbf{X}^{(p)}}^{(\mathbf{X}^{(p)},\mathbf{X}^{(p)})} = \epsilon_{\mathbf{X}^{(q)}}^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})}$.

Example 6.2. Let the MEV dependence function $D_{\mathbf{X}}$ defined by

$$D_{\mathbf{X}}(\mathbf{u}^{(3)}) = exp(-\sum_{j=1}^{3} z_j^{\theta})^{1/\theta}$$

where $z_j = -\log u_j$, $j = 1, \ldots, 3$, and $\theta \geq 1$ is constant.

Then, for $\mathbf{X}^{(p)} = (X_1, X_2)$ and $\mathbf{X}^{(q)} = X_3$, we have

$$\epsilon^{\mathbf{X}^{(p)}} = \lim_{u \to 1} \frac{2^{1/\theta} \log u}{\log u} = 2^{1/\theta}, \quad \epsilon^{\mathbf{X}^{(q)}} = 1, \quad \epsilon^{\mathbf{X}} = 3^{1/\theta}$$

and

$$\epsilon^{(\mathbf{X}^{(p)},\mathbf{X}^{(q)})} = \frac{3^{1/\theta}}{2^{1/\theta} + 1}.$$

Moreover, it holds $\alpha^{(p)} = -1 + \frac{2}{2^{1/\theta}}$, $\alpha^{(q)} = 1$ and $\beta^{(p,q)} = -1 + \frac{2+2^{1/\theta}}{3^{1/\theta}}$. Then, for $\theta > 1$, we find $\lambda_{\mathbf{X}^{(p)}}^{\mathbf{X}^{(q)}} = \frac{1-2\times 2^{1/\theta}+3^{1/\theta}}{2-2^{1/\theta}}$. For $\theta = 1$ it holds $\lambda_{\mathbf{X}^{(p)}}^{\mathbf{X}^{(q)}} = 1$ as expected.

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