A Note on Second Order Conditions in Extremes: Linking General and Heavy Tails Conditions*

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Abstract. Second order conditions ruling the rate of convergence towards zero in any first order condition involving regular variation that assures a unified extreme value limiting distribution function for the sequence of maximum values, linearly normalized, have appeared in several contexts whenever whenever researchers are working with heavy tails, i.e., with an extreme value index $\gamma > 0$ or with a general tail $\gamma \in \mathbb{R}$. In this paper we shall clarify the link between the second order parameters, say $\tilde{\rho}$ and ρ that have appeared in the two above mentioned set-ups. We illustrate the theory with some examples and, for heavy tails, we provide a link to a third order framework.

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1 Introduction

Let $X_1, X_2, \cdots, X_n$ be an independent, identically distributed (i.i.d.) sample from an unknown distribution function (d.f.) F. It is well-known from Gnedenko's seminal work (Gnedenko, 1943) that if there exist normalizing constants $a_n > 0$, $b_n \in \mathbb{R}$ and a non-degenerate d.f. $G(x)$ such that, for all x,

$$\lim_{n\to\infty} P\left\{a_n^{-1}\left(\max\left(X_1, \cdots, X_n\right) - b_n\right) \le x\right\} = G(x), \tag{1.1}$$

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G is, up to scale and location, an *Extreme Value* d.f., dependent on a shape parameter $\gamma \in \mathbb{R}$, and given by

$$EV_\gamma(x) := \begin{cases} \exp(-(1+\gamma\, x)^{-1/\gamma}),\ 1+\gamma\, x>0 & \text{if} \quad \gamma \neq 0 \\ \exp(-\exp(-x)),\ x \in \mathbb{R} & \text{if} \quad \gamma = 0 \end{cases}. \tag{1.2}$$

We then say that F is in the domain of attraction for maxima of the d.f. EV_γ in (1.2) and write $F \in \mathcal{D}_{\mathcal{M}}(EV_\gamma)$ for some $\gamma \in \mathbb{R}$.

2 First and second order conditions

2.1 Heavy tails ($\gamma > 0$)

The most typical first order condition for heavy tails, i.e., for the case $\gamma > 0$ in (1.2), comes also from Gnedenko (1943). Let us denote by RV_α the class of regularly varying functions with an index of regular variation α, i.e., positive measurable functions g such that $\lim_{t\to\infty} g(tx)/g(t) = x^\alpha$ for all $x > 0$. Then, for $\gamma > 0$,

$$F \in \mathcal{D}_{\mathcal{M}}(EV_\gamma) \quad \Longleftrightarrow \quad \overline{F} = 1 - F \in RV_{-1/\gamma}. \tag{2.1}$$

Equivalently, and with U standing for a quantile type function associated to F and defined by $U(t) := \big(1/(1-F)\big)^{\leftarrow}(t) = \inf\left\{x : F(x) \geq 1 - \frac{1}{t}\right\}$, de Haan (1970) established that

$$F \in \mathcal{D}_{\mathcal{M}}(EV_\gamma) \quad \Longleftrightarrow \quad U \in RV_\gamma. \tag{2.2}$$

To measure the rate of convergence in (2.2), it is then sensible to consider one of the following conditions:

$$\lim_{t\to\infty} \frac{\frac{U(tx)}{U(t)} - x^\gamma}{\widetilde{A}(t)} = x^\gamma\ \frac{x^{\tilde\rho}-1}{\tilde\rho} \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{\widetilde{A}(t)} = \frac{x^{\tilde\rho}-1}{\tilde\rho}, \tag{2.3}$$

for all $x > 0$, where $\tilde\rho \leq 0$ is a *second order* parameter controlling the speed of convergence of maximum values, linearly normalized, towards the limit law in (1.2) pertaining to $\gamma > 0$. Under these circumstances, we say that the function U is of *regular variation of second order*, and use the notation $U \in 2RV(\gamma, \tilde\rho)$. We remark that $|\widetilde{A}| \in RV_{\tilde\rho}$.

2.2 A general tail ($\gamma \in \mathbb{R}$)

The following *extended regular variation* property (de Haan, 1984), denoted ERV_γ, is a well-known necessary and sufficient condition for $F \in \mathcal{D}_{\mathcal{M}}(EV_\gamma)$:

$$\lim_{t\to\infty} \frac{U(tx) - U(t)}{a(t)} = \begin{cases} \frac{x^\gamma - 1}{\gamma} & \text{if} \quad \gamma \neq 0 \\ \ln x & \text{if} \quad \gamma = 0 \end{cases}, \tag{2.4}$$

for every $x > 0$ and some positive measurable function a. For the case $\gamma > 0$ we see easily from (2.2) that we can choose $a(t) = \gamma U(t)$.

Apart from the first order condition in (2.4), we shall consider the most common second order condition, specifying the rate of convergence in (2.4). We shall assume the existence of a function A, possibly not changing in sign and tending to zero as $t \to \infty$, such that

$$\lim_{t\to\infty} \frac{\frac{U(tx)-U(t)}{a(t)} - \frac{x^\gamma - 1}{\gamma}}{A(t)} = H_{\gamma,\rho}(x) := \frac{1}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^\gamma - 1}{\gamma}\right) \tag{2.5}$$

for all $x > 0$, where $\rho \le 0$ is also a *second order* parameter controlling the speed of convergence of maximum values, linearly normalized, towards the limit law in (1.2), for a general $\gamma \in \mathbb{R}$. We then say that the function U is of *second order extended regular variation, and use the notation* $U \in 2ERV_{\gamma,\rho}$. We remark that $|A| \in RV_\rho$. For a large variety of models we have $\rho < 0$ thus making sensible to simplify (2.5). We state without proof:

Proposition 2.1. *Let us assume that there exist $a(\cdot)$ and $A(\cdot)$ such that* (2.5) *holds, with $\rho < 0$. Then, there exist $a_0(\cdot)$ and $A_0(\cdot)$ such that*

$$\lim_{t\to\infty} \frac{\frac{U(tx)-U(t)}{a_0(t)} - \frac{x^\gamma - 1}{\gamma}}{A_0(t)} = \frac{x^{\gamma+\rho}-1}{\gamma+\rho} \tag{2.6}$$

with

$$A_0(t) = A(t)/\rho, \qquad a_0(t) = a(t)(1 - A_0(t)). \tag{2.7}$$

From Theorem A in Draisma de Haan, Peng and Pereira (1999), with slight additions in Ferreira, de Haan and Peng (2003) and in de Haan and Ferreira (2006), we state the following:

Theorem 2.1. *Suppose the right endpoint $x^F = U(\infty) > 0$ and there exist $a(\cdot)$ and $A(\cdot)$ such that* (2.5) *holds, with $\rho \le 0$, $\gamma \ne \rho$. Define*

$$\overline{A}(t) := \left(\frac{a(t)}{U(t)} - \gamma_+\right), \qquad \gamma_+ := \max(0,\ \gamma). \tag{2.8}$$

Then for $\gamma + \rho < 0$

$$l := \lim_{t\to\infty}\left(U(t) - \frac{a(t)}{\gamma}\right) \textit{ exists finite} \tag{2.9}$$

and the following holds

$$\overline{A}(t) \underset{t\to\infty}{\longrightarrow} 0 \qquad \textit{and} \qquad \frac{\overline{A}(t)}{A(t)} \underset{t\to\infty}{\longrightarrow} c,$$

with

$$c = \begin{cases} 0 & \textit{if} \quad \gamma < \rho \le 0 \\ \frac{\gamma}{\gamma+\rho} & \textit{if} \quad 0 \le -\rho < \gamma \ \textit{or} \ (0 < \gamma < -\rho \ \textit{and} \ l = 0) \\ \pm\infty & \textit{if} \quad \gamma + \rho = 0 \ \textit{or} \ (0 < \gamma < -\rho \ \textit{and} \ l \ne 0) \ \textit{or} \ \rho < \gamma \le 0. \end{cases} \tag{2.10}$$

3 The link between the second order condition for a heavy and for a general tail

The following results hold with any measurable (eventually) positive function U.

Lemma 3.1. *If* (2.4) *holds for some* $\gamma \in \mathbb{R}$, *then the auxiliary function* $a(t)$ *in* (2.4) *is of regular variation at infinity with index* γ, *i.e.,* $a \in RV_\gamma$ *and*

$$\lim_{t\to\infty} \frac{a(t)}{U(t)} = \gamma_+ := \max(0,\gamma).$$

Moreover, if $\gamma > 0$, *both functions* a *and* U *belong to* RV_γ; *if* $\gamma < 0$, *then* $U(\infty) < \infty$, $\lim_{t\to\infty} a(t)/(x^F - U(t)) = -\gamma$ *and* $U(\infty) - U(t) \in RV_\gamma$.

Furthermore, with $\gamma_- := \min(\gamma, 0)$, *and provided that* $U(\infty) > 0$,

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t)}{a(t)/U(t)} = \frac{x^{\gamma_-} - 1}{\gamma_-}, \quad \textit{for every } x > 0. \tag{3.1}$$

Proof. The first part of the lemma comes from Theorems 1.9 and 1.10 in Geluk and de Haan (1987). The limit in (3.1) follows easily when we distinguish between the cases $\gamma > 0$ and $\gamma \le 0$. ■

For the derivation of asymptotic properties of semi-parametric estimators of γ, a topic out of the scope of this paper, it is important to know, for all $x > 0$, not only the rate of convergence of $\ln U(tx) - \ln U(t)$, but also of $U(tx)/U(t)$ and of $U(t)/U(tx)$, as $t \to \infty$. We shall now see in more detail for the different relevant subspaces of the semi-plane $(\gamma, \rho) \in \mathbb{R} \times \mathbb{R}_0^-$, the limiting behaviour, as $t \to \infty$, of $U(tx)/U(t)$ and $U(t)/U(tx)$. The limit behavior of $\ln U(tx) - \ln U(t)$ has been analyzed e.g. in Appendix B of de Haan and Ferreira (2006).

Lemma 3.2. *Assume that* (2.5) *holds, i.e.,* $U \in 2ERV_{\gamma,\rho}$. *Then, we may write*

$$\frac{U(tx)}{U(t)} = x^{\gamma_+} + \overline{A}(t)\Big[\frac{x^\gamma - 1}{\gamma} + A(t)\,\overline{a}(x,t;\gamma,\rho)\big(1+o(1)\big)\Big], \tag{3.2}$$

where

$$\overline{a}(x,t;\gamma,\rho) = \begin{cases} \frac{\ln^2 x}{2} & \textit{if} \;\; \gamma=\rho=0 \\ \frac{1}{\gamma}\left(x^\gamma \ln x - \frac{x^\gamma-1}{\gamma}\right) & \textit{if} \;\; \gamma<\rho=0 \\ \frac{1}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^\gamma-1}{\gamma}\right) & \textit{if} \;\; \gamma\le 0,\ \rho<0 \\ \frac{\gamma}{\rho\,\overline{A}(t)}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^\gamma-1}{\gamma}\right) & \textit{if} \;\; \gamma>0,\ \rho<0 \\ \frac{1}{\overline{A}(t)}\left(x^\gamma \ln x - \frac{x^\gamma-1}{\gamma}\right) & \textit{if} \;\; \rho=0<\gamma \end{cases}.$$

Proof. Directly from (2.5), we get

$$\frac{U(tx)}{U(t)} - 1 = \frac{a(t)}{U(t)} \left\{ \frac{x^\gamma - 1}{\gamma} + \frac{A(t)}{\rho} \left(\frac{x^{\gamma+\rho} - 1}{\gamma + \rho} - \frac{x^\gamma - 1}{\gamma} \right) (1 + o(1)) \right\}.$$

With the notation in (2.8), i.e., $a(t)/U(t) = \gamma_+ + \overline{A}(t)$, we may write

$$\frac{U(tx)}{U(t)} - 1 = \gamma_+ \left(\frac{x^\gamma - 1}{\gamma} \right) + \overline{A}(t) \left(\frac{x^\gamma - 1}{\gamma} \right) + \frac{A(t)}{\rho} \left(\frac{x^{\gamma+\rho} - 1}{\gamma + \rho} - \frac{x^\gamma - 1}{\gamma} \right) \left(\gamma_+ + \overline{A}(t) \right) (1 + o(1))$$

and (3.2) follows for any $\rho < 0$.

If $\rho = 0$ and $\gamma \neq 0$, then, also directly from (2.5), and by continuity arguments,

$$\frac{U(tx)}{U(t)} - 1 = \frac{a(t)}{U(t)} \left\{ \frac{x^\gamma - 1}{\gamma} + \frac{A(t)}{\gamma} \left(x^\gamma \ln x - \frac{x^\gamma - 1}{\gamma} \right) (1 + o(1)) \right\},$$

and things work as before, with $A(t)/\rho$ replaced by $A(t)/\gamma$ and $\frac{x^{\gamma+\rho}-1}{\gamma+\rho}$ replaced by $x^\gamma \ln x$. The case $\gamma = \rho = 0$ comes again directly from (2.5) and by continuity arguments. ■

Theorem 3.1. *Let $U \in ERV_{\gamma,\rho}$ as introduced in (2.5). Let c be the limit in (2.10).*

(i) If $\gamma > 0$,

$$\lim_{t\to\infty} \frac{\frac{U(t)}{U(tx)} - x^{-\gamma}}{\widetilde{A}(t)} = K_{\gamma,\rho}(x) := \begin{cases} -x^{-\gamma}\ \frac{x^\rho - 1}{\rho} & \text{if} \quad c = \frac{\gamma}{\gamma+\rho} \\ -x^{-\gamma}\ \frac{x^{-\gamma}-1}{-\gamma} & \text{if} \quad c = \pm\infty \end{cases}, \tag{3.3}$$

for all $x > 0$, where, with $\overline{A}(t)$ given in (2.8),

$$\widetilde{A}(t) := \begin{cases} \frac{\gamma\ A(t)}{\gamma+\rho} & \text{if} \quad c = \frac{\gamma}{\gamma+\rho} \\ \overline{A}(t) & \text{if} \quad c = \pm\infty \end{cases}, \tag{3.4}$$

Necessarily, $|\widetilde{A}| \in RV_{\widetilde{\rho}}$, with

$$\widetilde{\rho} = \begin{cases} \rho & \text{if} \quad c = \frac{\gamma}{\gamma+\rho} \\ -\gamma & \text{if} \quad c = \pm\infty \end{cases}. \tag{3.5}$$

(ii) If $\gamma \leq 0$, we need further to assume that $\gamma \neq \rho$. Then,

$$\lim_{t\to\infty} \frac{\frac{U(t)}{a^*(t)} \left(1 - \frac{U(t)}{U(tx)} \right) - \frac{x^\gamma - 1}{\gamma}}{A^*(t)} = K^*_{\gamma,\ \rho}(x) = \begin{cases} x^\gamma \ln x & \text{if} \quad \gamma < \rho = 0 \\ \frac{x^{\gamma+\rho}-1}{\gamma+\rho} & \text{if} \quad \gamma < \rho < 0 \\ \frac{x^{2\gamma}-1}{2\gamma} & \text{if} \quad \rho < \gamma < 0 \\ \ln^2 x & \text{if} \quad \rho < \gamma = 0 \end{cases}, \tag{3.6}$$

where

$$A^*(t) = \begin{cases} \frac{A(t)}{\gamma} & \text{if} \quad \gamma < \rho = 0 \\ \frac{A(t)}{\rho} & \text{if} \quad \gamma < \rho < 0 \\ -\frac{2\ \overline{A}(t)}{\gamma} & \text{if} \quad \rho < \gamma < 0 \\ -\overline{A}(t) & \text{if} \quad \rho < \gamma = 0 \end{cases}, \tag{3.7}$$

and

$$a^*(t) = \begin{cases} a(t) & \text{if} \qquad \rho < \gamma = 0 \\ a(t)\,(1 - A^*(t)) & \text{otherwise} \end{cases}. \tag{3.8}$$

Necessarily, $|A^*| \in RV_{\rho^*}$, *with*

$$\rho^* = \begin{cases} \rho & \text{if} \quad \gamma < \rho \le 0 \\ \gamma & \text{if} \quad \rho < \gamma \le 0 \end{cases}. \tag{3.9}$$

Proof. We shall consider the cases (i) and (ii) separately.
Case (i) : If $\gamma > 0$, $\rho < 0$ and (2.5) holds, i.e., $U \in 2ERV_{\gamma,\rho}$, we have from (3.2),

$$\tfrac{U(tx)}{U(t)} - x^\gamma = \overline{A}(t)\left(\tfrac{x^\gamma - 1}{\gamma}\right) + \tfrac{\gamma\ A(t)}{\rho}\left(\tfrac{x^{\gamma+\rho}-1}{\gamma+\rho} - \tfrac{x^\gamma - 1}{\gamma}\right) + o(A(t)).$$

If $c = \pm\infty$, then $A(t) = o\left(\overline{A}(t)\right)$,

$$\tfrac{U(tx)}{U(t)} - x^\gamma = x^\gamma\left(\tfrac{x^{-\gamma}-1}{-\gamma}\right)\ \overline{A}(t) + o\left(\overline{A}(t)\right), \quad \text{and} \quad \frac{\frac{U(tx)}{U(t)} - x^\gamma}{\overline{A}(t)} \underset{t\to\infty}{\longrightarrow} x^\gamma\left(\tfrac{x^{-\gamma}-1}{-\gamma}\right).$$

If $c = \gamma/(\gamma+\rho)$, we get $\overline{A}(t) = \frac{\gamma\ A(t)}{\gamma+\rho}(1 + o(1))$. Since in this region $\gamma \neq -\rho$, we may further write

$$\begin{aligned} \tfrac{U(tx)}{U(t)} - x^\gamma &= x^\gamma\left(\overline{A}(t)\left(\tfrac{x^{-\gamma}-1}{-\gamma}\right) + \tfrac{\gamma\ A(t)}{\gamma+\rho}\left(\tfrac{x^\rho - 1}{\rho} - \tfrac{x^{-\gamma}-1}{-\gamma}\right) + o(A(t))\right) \\ &= \frac{\gamma\ A(t)}{\gamma+\rho}\ x^\gamma\left(\frac{x^\rho - 1}{\rho}\right) + o(A(t)). \end{aligned}$$

Consequently,

$$\lim_{t\to\infty} \frac{\frac{U(tx)}{U(t)} - x^\gamma}{\widetilde{A}(t)} = \begin{cases} x^\gamma\ \frac{x^\rho - 1}{\rho} & \text{if} \quad c = \frac{\gamma}{\gamma+\rho} \\ x^\gamma\ \frac{x^{-\gamma}-1}{-\gamma} & \text{if} \quad c = \pm\infty \end{cases} = -x^{2\gamma} K_{\gamma,\rho}(x), \tag{3.10}$$

with $K_{\gamma,\rho}(x)$ and $\widetilde{A}(t)$ defined in (3.3) and (3.4), respectively. Finally, (3.10), together with the fact that

$$\frac{U(tx)}{U(t)} - x^\gamma = -x^\gamma\ \frac{U(tx)}{U(t)}\left(\frac{U(t)}{U(tx)} - x^{-\gamma}\right) = -x^{2\gamma}\left(\frac{U(t)}{U(tx)} - x^{-\gamma}\right)(1 + o(1)),$$

leads us to the limit in (3.3), with $\widetilde{A}(t)$ and $\widetilde{\rho}$ given in (3.4) and (3.5), respectively.

If $\gamma > 0$ and $\rho = 0$, we get, again from (3.2),

$$\begin{aligned}\frac{U(tx)}{U(t)} - x^{\gamma} &= \overline{A}(t)\ \left(\tfrac{x^{\gamma}-1}{\gamma}\right) + A(t)\left(x^{\gamma}\ \ln x - \tfrac{x^{\gamma}-1}{\gamma}\right) + o(A(t)) \\ &= x^{\gamma}\left(\overline{A}(t)\ \left(\tfrac{x^{-\gamma}-1}{-\gamma}\right) + A(t)\left(\ln x - \tfrac{x^{-\gamma}-1}{-\gamma}\right) + o(A(t))\right).\end{aligned}$$

But if $\gamma > 0$ and $\rho = 0$, then $c = \gamma/(\gamma+\rho) = 1$, $\overline{A}(t) = A(t) + o(A(t))$, and

$$\frac{U(tx)}{U(t)} - x^{\gamma} = A(t)\ x^{\gamma}\ln x + o(A(t)).$$

Consequently, (3.3) holds, with $\widetilde{A}(t) = A(t) \equiv \gamma\ A(t)/(\gamma+\rho)$ and $\widetilde{\rho} = \rho = 0$.

Case (*ii*): If $\gamma < \rho = 0$, we get, again from (3.2),

$$\begin{aligned}\frac{U(t)}{a(t)}\left(1 - \frac{U(t)}{U(tx)}\right) &= \frac{x^{\gamma}-1}{\gamma} + \frac{A(t)}{\gamma}\left(x^{\gamma}\ln x - \frac{x^{\gamma}-1}{\gamma}\right) + o(A(t)) \\ &= \frac{x^{\gamma}-1}{\gamma}\left(1 - \frac{A(t)}{\gamma}\right) + \frac{A(t)}{\gamma}\ x^{\gamma}\ln x + o(A(t))\end{aligned}$$

and with $a^{*}(t) = a(t)\left(1 - \frac{A(t)}{\gamma}\right)$,

$$\frac{U(t)}{a^{*}(t)}\left(1 - \frac{U(t)}{U(tx)}\right) = \frac{x^{\gamma}-1}{\gamma} + \frac{A(t)}{\gamma}\ x^{\gamma}\ln x + o(A(t)).$$

Consequently, (3.6), (3.7), (3.8) and (3.9) follow in this region of the (γ,ρ)-plane.

If $\gamma < \rho < 0$, $a(t)/U(t) \equiv \overline{A}(t) = o(A(t))$, and again from (3.2),

$$\begin{aligned}\frac{U(t)}{a(t)}\left(1 - \frac{U(t)}{U(tx)}\right) &= \frac{x^{\gamma}-1}{\gamma} + \frac{A(t)}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^{\gamma}-1}{\gamma}\right) + o(A(t)) \\ &= \frac{x^{\gamma}-1}{\gamma}\left(1 - \frac{A(t)}{\rho}\right) + \frac{A(t)}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho}\right) + o(A(t))\end{aligned}$$

and with $a^{*}(t) = a(t)\left(1 - \frac{A(t)}{\rho}\right)$,

$$\frac{U(t)}{a^{*}(t)}\left(1 - \frac{U(t)}{U(tx)}\right) = \frac{x^{\gamma}-1}{\gamma} + \frac{A(t)}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho}\right) + o(A(t)),$$

and the results in the proposition hold.
If $\rho < \gamma \le 0$, $A(t) = o\left(\overline{A}(t)\right)$, and also from (3.2), we get

$$\frac{U(t)}{U(tx)} = 1 - \overline{A}(t)\left(\tfrac{x^{\gamma}-1}{\gamma}\right) + \overline{A}^{2}(t)\left(\tfrac{x^{\gamma}-1}{\gamma}\right)^{2}(1+o(1)). \tag{3.11}$$

Consequently, for $\gamma < 0$, since $\left(\frac{x^\gamma-1}{\gamma}\right)^2 = \frac{2}{\gamma}\left(\frac{x^{2\gamma}-1}{2\gamma} - \frac{x^\gamma-1}{\gamma}\right)$

$$\begin{aligned}\frac{U(t)}{a(t)}\left(1-\frac{U(t)}{U(tx)}\right) &= \tfrac{x^\gamma-1}{\gamma} - \frac{2\,\overline{A}(t)}{\gamma}\left(\tfrac{x^{2\gamma}-1}{2\gamma} - \tfrac{x^\gamma-1}{\gamma}\right)(1+o(1))\\ &= \tfrac{x^\gamma-1}{\gamma}\left(1+\frac{2\,\overline{A}(t)}{\gamma}\right) - \frac{2\,\overline{A}(t)}{\gamma}\left(\tfrac{x^{2\gamma}-1}{2\gamma}\right)(1+o(1)),\end{aligned}$$

and with $a^*(t) = a(t)\left(1+\frac{2\,\overline{A}(t)}{\gamma}\right)$,

$$\frac{U(t)}{a^*(t)}\left(1-\frac{U(t)}{U(tx)}\right) = \tfrac{x^\gamma-1}{\gamma} - \frac{2\,\overline{A}(t)}{\gamma}\left(\tfrac{x^{2\gamma}-1}{2\gamma}\right)(1+o(1)),$$

and the results in the proposition follow.
If $\rho < \gamma = 0$, then from (3.11), we get

$$\frac{U(t)}{a(t)}\left(1-\frac{U(t)}{U(tx)}\right) = \ln x - \overline{A}(t)\ \ln^2 x(1+o(1)),$$

and the result in the proposition follows as well. ∎

Corollary 3.1. *Under the conditions and notations of Proposition 2.1, and for* $\gamma > 0$,

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \gamma\ln x}{\widetilde{A}(t)} = \widetilde{K}_{\gamma,\rho}(x) := \begin{cases} \frac{x^\rho-1}{\rho} & \textit{if}\quad c = \frac{\gamma}{\gamma+\rho} \\ \frac{x^{-\gamma}-1}{-\gamma} & \textit{if}\quad c = \pm\infty \end{cases}, \tag{3.12}$$

for every $x > 0$, *and with* $\widetilde{A}$ *provided in* (3.4).

Proof. The proof follows immediately from relation (3.3). ∎

Remark 3.1. *Note that the second order condition in* (3.12) *is the usual second order condition for heavy tails, i.e., the second order condition provided in* (2.3).

Remark 3.2. *Note next that the region* $\{(\gamma,\rho) : 0 < \gamma < -\rho \text{ and } l \neq 0\}$ *in the* (γ,ρ)*-plane, jointly with the line* $\rho = -\gamma$, *are transformed in the line* $\tilde{\rho} = -\gamma$ *in the* $(\gamma,\tilde{\rho})$*-plane. There we have* $c = \pm\infty$. *Outside that line we have* $c = \gamma/(\gamma+\tilde{\rho}) = \gamma/(\gamma+\rho)$.

Remark 3.3. *For* $\gamma > 0$, *the rate of convergence in* (3.1), *i.e., the rate of convergence of* $(\ln U(tx) - \ln U(t))/(a(t)/U(t)) - \ln x$ *towards zero, is measured by* $\widetilde{A}(t)$ *in* (3.4) *only if* $\rho \neq 0$. *If* $\rho = 0$, *the rate of convergence in* (3.1) *can be of a smaller order than* $\widetilde{A}(t)$ *as may be seen in Example 4.1. For* $\gamma \leq 0$, *Lemma 3.2 gives rise to* (3.1) *in a similar way as it yields Corollary 3.1.*

4 Examples and some additional comments

Example 4.1. (A model with $\rho = 0$ and $\gamma > 0$). *Consider the model $U(t) = t^\gamma\,(1 + \ln t)$. Then*

$$U(tx) - U(t) = \gamma\ t^\gamma(\ln t + 1)\left(\frac{x^\gamma - 1}{\gamma} + \frac{x^\gamma \ln x}{\gamma(\ln t + 1)}\right), \quad x > 0,$$

i.e.,

$$\frac{\frac{U(tx)-U(t)}{\gamma\ t^\gamma(\ln t+1)} - \frac{x^\gamma-1}{\gamma}}{1/\left(\gamma(\ln t + 1)\right)} = x^\gamma \ln x.$$

Notice that $H_{\gamma,0}(x) = \gamma^{-1}\big(x^\gamma \ln x - (x^\gamma - 1)/\gamma\big)$, meaning that (2.5) is equivalent to

$$\frac{\frac{U(tx)-U(t)}{a(t)(1-A(t)/\gamma)} - \frac{x^\gamma-1}{\gamma}}{A(t)/\gamma} \underset{t\to\infty}{\longrightarrow} x^\gamma \ln x. \tag{4.1}$$

Consequently we should choose

$$A(t) = \frac{1}{\ln t + 1} \in RV_0, \quad a(t)\left(1 - \frac{1}{\gamma(\ln t + 1)}\right) = \gamma t^\gamma(\ln t + 1).$$

Theorem 2.1 yields $c = 1$ while Theorem 3.1 determines $\tilde\rho = 0$ and $\widetilde{A}(t) = A(t)$. Indeed, we have

$$\ln U(tx) - \ln U(t) - \gamma \ln x \ \ = \ \ \frac{\ln x}{\ln t + 1} + \frac{\ln^2 x}{2(\ln t + 1)^2} + o\Big(\frac{1}{\ln^2 t}\Big), \tag{4.2}$$

as $t \to \infty$, thus making suitable to take $A(t) = (\ln t + 1)^{-1}$ in the left hand side of

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{A(t)} \ \ = \ \ \ln x, \quad x > 0$$

and (3.12) holds in fact with $\widetilde{A}(t) = A(t)$. Furthermore, after a few manipulations of (4.2), we get

$$\frac{\frac{\ln U(tx)-\ln U(t)}{\gamma\left(1+\frac{1}{\gamma(\ln t+1)}\right)} - \ln x}{\frac{1}{2\ln^2 t}} \underset{t\to\infty}{\longrightarrow} \ln^2 x.$$

Therefore, the rate of convergence in (3.1) is of the order of $1/\ln^2 t = o(A(t))$, as mentioned in Remark 3.3.

Example 4.2. *For the Fréchet model, $F(x) = \exp\left(-x^{-1/\gamma}\right)$, $x \geq 0$ $(\gamma > 0)$, we get successively,*

$$\begin{aligned} U(t) \ &= \ \Big(-\ln\Big(1 - \frac{1}{t}\Big)\Big)^{-\gamma} \\ &= \ t^\gamma\Big(1 + \frac{1}{2\ t} + \frac{1}{3\ t^2} + o\left(t^{-2}\right)\Big)^{-\gamma} \\ &= \ t^\gamma\Big(1 - \frac{\gamma}{2\ t} + \frac{\gamma(3\gamma - 5)}{24\ t^2} + o\left(t^{-2}\right)\Big). \end{aligned}$$

Hence,

$$U(tx) - U(t) = \begin{cases} \gamma\, t^{\gamma} \left(\frac{x^{\gamma}-1}{\gamma} - \frac{\gamma-1}{2\ t} \left(\frac{x^{\gamma-1}-1}{\gamma-1} \right) + o(t^{-1}) \right) & \text{if} \quad \gamma \neq 1 \\ \\ t\left((x-1) - \frac{1}{12\ t^2} \left((x^{-1}-1) \right) + o(t^{-2}) \right) & \text{if} \quad \gamma = 1 \end{cases}.$$

If we make correspondence with condition (2.6), *we see that* $\rho = \begin{cases} -1 & \text{if} \quad \gamma \neq 1 \\ -2 & \text{if} \quad \gamma = 1 \end{cases}$. *Likewise,* (2.7) *can be set as*

$$a_0(t) = \gamma\ t^{\gamma} \quad \text{and} \quad A_0(t) = \begin{cases} \frac{1-\gamma}{2\ t} & \text{if} \quad \gamma \neq 1 \\ \frac{1}{12\ t^2} & \text{if} \quad \gamma = 1 \end{cases}.$$

According to Proposition 2.1, if we choose

$$A(t) = \rho A_0(t) = \begin{cases} \frac{\gamma-1}{2\ t} & \text{if} \quad \gamma \neq 1 \\ -\frac{1}{6\ t^2} & \text{if} \quad \gamma = 1 \end{cases}$$

and

$$a(t) = \gamma\ t^{\gamma}/(1 - A_0(t)) = \begin{cases} \frac{2\ \gamma\ t^{\gamma+1}}{2t+\gamma-1} & \text{if} \quad \gamma \neq 1 \\ \frac{12\ t^3}{12\ t^2-1} & \text{if} \quad \gamma = 1 \end{cases},$$

we get the limiting result in (2.5).

We will derive that (3.12) *holds true for* $\widetilde{A}(t) = \gamma/(2\ t)$, *with* $\tilde{\rho} = -1 \neq \rho = -2$ *for* $\gamma = 1$, *and* $\tilde{\rho} = -1 = \rho$ *for* $\gamma \neq 1$. *As seen before regarding the limit in* (2.5), *we have whenever* $\gamma \neq 1$,

$$\begin{aligned} U(t) &= t^{\gamma}\Big(1 - \frac{\gamma}{2\ t} + \frac{\gamma(3\gamma-5)}{24\ t^2} + o\left(t^{-2}\right)\Big); \\ a(t) &= 2\ \gamma\ t^{\gamma+1}/(2t+\gamma+\rho); \\ A(t) &= -\rho(\gamma+\rho)/(2\ t). \end{aligned}$$

Then,

$$\begin{aligned} \overline{A}(t) = \frac{a(t)}{U(t)} - \gamma &= \frac{2\ \gamma\ t}{2\ t+\gamma+\rho}\Big(1 + \frac{\gamma}{2\ t} - \frac{\gamma(3\gamma-5)}{24\ t^2} + \frac{\gamma^2}{4t^2} + o\left(t^{-2}\right)\Big) - \gamma \\ &= \frac{2\ \gamma\ t(2\ t+\gamma)}{2\ t(2\ t+\gamma+\rho)} - \gamma - \frac{2\ \gamma^2\ t(9\gamma-5)}{24\ t^2(2\ t+\gamma+\rho)} + o\left(t^{-2}\right) \\ &= -\frac{\gamma\ \rho}{2t+\gamma+\rho}\Big(1 + \frac{\gamma(9\gamma-5)}{12\ \rho\ t} + o(t^{-1})\Big) \underset{t\to\infty}{\longrightarrow} 0, \end{aligned}$$

and

$$\frac{\overline{A}(t)}{A(t)} = \frac{2\ \gamma\ t}{(\gamma+\rho)(2t+\gamma+\rho)}\Big(1 + \frac{\gamma(9\gamma-5)}{12\ t} + o(t^{-1})\Big) \underset{t\to\infty}{\longrightarrow} \frac{\gamma}{\gamma+\rho}.$$

Let us think on

$$U(t)-\frac{a(t)}{\gamma} \;=\; -\frac{U(t)}{\gamma}\overline{A}(t) \;=\; t^\gamma\Big(\frac{2\;\rho\;t-\gamma(\gamma+\rho)}{2\;t(2t+\gamma+\rho)}+\frac{\gamma(3\gamma-5)}{24\;t^2}+o\left(t^{-2}\right)\Big)$$
$$\underset{t\to\infty}{\longrightarrow}\; 0=:l, \qquad \textit{if}\quad 0<\gamma<1.$$

Hence, we conclude that $c=\gamma/(\gamma+\rho)$, *for* $\gamma\neq 1$.

If we consider the case $\gamma=1$,

$$\frac{a(t)}{U(t)} \;=\; \frac{12\;t^2}{12\;t^2-1}\Big(1+\frac{1}{2\;t}+\frac{1}{12\;t^2}+\frac{1}{4t^2}+o\left(t^{-2}\right)\Big)$$
$$=\; 1+\frac{1}{2t}+\frac{5}{12t^2}+o(t^{-2}).$$

Consequently, and as was expected from Theorem 2.1,

$$\overline{A}(t)=\frac{a(t)}{U(t)}-1 \;=\; \frac{1}{2\;t}\Big(1+\frac{5}{6t}+o\left(t^{-1}\right)\Big)\;\underset{t\to\infty}{\longrightarrow}\;0,$$

$$\frac{\overline{A}(t)}{A(t)} \;=\; -3t\Big(1+\frac{5}{6t}+o\left(t^{-1}\right)\Big)\;\underset{t\to\infty}{\longrightarrow}-\infty,\;\; \textit{i.e.},\;\; c=-\infty$$

and

$$U(t)-a(t) \;=\; -\frac{1}{2}-\frac{t}{12\;t^2-1}-\frac{1}{12\;t}+o\left(t^{-1}\right)\;\underset{t\to\infty}{\longrightarrow}-\frac{1}{2}=l.$$

Since this limit l *is different from zero and* $\gamma=1<-\rho=2$, *we indeed expected to have* $c=\pm\infty$, *as actually happens. Now, from Theorem 3.1,* $\tilde\rho=-\gamma=-1$ *and we may choose*

$$\widetilde{A}(t)=\overline{A}(t)=\frac{1}{2\;t}\Big(1+\frac{5}{6t}+o\left(t^{-1}\right)\Big),$$

or more simply $\widetilde{A}(t)=1/(2t)$. *Indeed, and as mentioned before for* $\gamma=1$, (3.12) *holds true with* $\widetilde{A}(t)=\gamma/(2\;t)$ *and* $\tilde\rho=-1\neq\rho=-2$.

Example 4.3. *Consider the extreme value model with d.f.* $EV_\gamma(x)=\exp\left(-(1+\gamma\;x)^{-1/\gamma}\right)$, $1+\gamma x>0$, $\gamma\in\mathbb{R}$. *For this model,*

$$\begin{aligned}
U(t) \;&=\; \frac{\left(-\ln\left(1-\frac{1}{t}\right)\right)^{-\gamma}-1}{\gamma}\\
&=\; \frac{t^\gamma}{\gamma}\left(1-t^{-\gamma}-\frac{\gamma}{2t}+\frac{\gamma(3\gamma-5)}{24t^2}+o\left(t^{-2}\right)\right)\\
&=\; \begin{cases}
-\frac{1}{\gamma}\Big(1-t^\gamma+\frac{\gamma t^{\gamma-1}}{2}+o(t^{\gamma-1})\Big) & \textit{if}\quad \gamma<0\\
\ln t-\frac{1}{2t}+o(t^{-1}) & \textit{if}\quad \gamma=0\\
\frac{t^\gamma}{\gamma}\left(1-t^{-\gamma}-\frac{\gamma}{2t}+o\left(t^{-1}\right)\right) & \textit{if}\quad 0<\gamma<1\\
\frac{t^\gamma}{\gamma}\left(1-\frac{3}{2t}-\frac{1}{12t^2}+o\left(t^{-2}\right)\right) & \textit{if}\quad \gamma=1\\
\frac{t^\gamma}{\gamma}\left(1-\frac{\gamma}{2t}+o\left(t^{-1}\right)\right) & \textit{if}\quad \gamma>1
\end{cases}.
\end{aligned}$$

Then

$$U(tx)-U(t)=\begin{cases} t^{\gamma}\left(\frac{x^{\gamma}-1}{\gamma}-\frac{\gamma-1}{2t}\left(\frac{x^{\gamma-1}-1}{\gamma-1}\right)+o\left(t^{-1}\right)\right) & \textit{if} \quad \gamma\neq 1 \\ \\ t^{\gamma}\left(\frac{x^{\gamma}-1}{\gamma}+\frac{\gamma-2}{12\ t^2}\left(\frac{x^{\gamma-2}-1}{\gamma-2}\right)+o\left(t^{-2}\right)\right) & \textit{if} \quad \gamma=1 \end{cases},$$

i.e., we may choose, in (2.6),

$$a_0(t)=t^{\gamma} \quad \textit{and } A_0(t)=\begin{cases} -\frac{\gamma-1}{2\ t} & \textit{if} \quad \gamma\neq 1 \\ -\frac{1}{12\ t^2} & \textit{if} \quad \gamma=1 \end{cases}, \quad \textit{with} \quad \rho=\begin{cases} -1 & \textit{if} \quad \gamma\neq 1 \\ -2 & \textit{if} \quad \gamma=1 \end{cases}$$

Since

$$1-A_0(t)=\begin{cases} \frac{2t+\gamma-1}{2\ t} & \textit{if} \quad \gamma\neq 1 \\ \frac{12\ t^2-1}{12\ t^2} & \textit{if} \quad \gamma=1 \end{cases},$$

we get, from (2.7),

$$a(t)=\frac{a_0(t)}{1-A_0(t)}=\begin{cases} \frac{2\ t^{\gamma+1}}{2t+\gamma-1} & \textit{if} \quad \gamma\neq 1 \\ \frac{12\ t^3}{12\ t^2-1} & \textit{if} \quad \gamma=1 \end{cases}$$

and

$$A(t)=\rho\ A_0(t)=\begin{cases} \frac{\gamma-1}{2\ t} & \textit{if} \quad \gamma\neq 1 \\ \frac{1}{6\ t^2} & \textit{if} \quad \gamma=1. \end{cases}$$

Then

$$\frac{a(t)}{U(t)}=\begin{cases} -\gamma t^{\gamma}\big(1+(\frac{1-\gamma}{2t}+t^{\gamma})(1+o(1))\big) & \textit{if} \quad \gamma<0 \\ \frac{1}{\ln t}\big(1+\frac{1}{2t}+o(t^{-1})\big) & \textit{if} \quad \gamma=0 \\ \gamma\big(1+t^{-\gamma}+o\left(t^{-\gamma}\right)\big) & \textit{if} \quad 0<\gamma<1 \\ 1+\frac{3}{2t}+o\left(t^{-1}\right) & \textit{if} \quad \gamma=1 \\ \gamma\big(1+\frac{1}{2t}+o\left(t^{-1}\right)\big) & \textit{if} \quad \gamma>1 \end{cases},$$

and consequently,

$$\overline{A}(t)=\frac{a(t)}{U(t)}-\gamma_{+}=\begin{cases} -\gamma t^{\gamma}(1+o(1)) & \textit{if} \quad \gamma<0 \\ \frac{1}{\ln t}(1+o(1)) & \textit{if} \quad \gamma=0 \\ \gamma\ t^{-\gamma}+o\left(t^{-\gamma}\right) & \textit{if} \quad 0<\gamma<1 \\ \frac{3}{2t}+o\left(t^{-1}\right) & \textit{if} \quad \gamma=1 \\ \frac{\gamma}{2\ t}\ +o\left(t^{-1}\right) & \textit{if} \quad \gamma>1 \end{cases}.$$

Then

$$\frac{\overline{A}(t)}{A(t)} = \begin{cases} -\frac{2\gamma t^{\gamma+1}}{\gamma-1}(1+o(1)) & \text{if } \gamma<0 \\ -\frac{2t}{\ln t}(1+o(1)) & \text{if } \gamma=0 \\ \frac{2\gamma}{\gamma-1}t^{1-\gamma}(1+o(1)) & \text{if } 0<\gamma<1 \\ -9t(1+o(1)) & \text{if } \gamma=1 \\ \frac{\gamma}{\gamma-1}(1+o(1)) & \text{if } \gamma>1 \end{cases}$$

$$\underset{t\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } \gamma<-1 \\ -\infty & \text{if } -1<\gamma\le 1 \\ \frac{\gamma}{\gamma-1}=\frac{\gamma}{\gamma+\rho} & \text{if } \gamma>1 \end{cases} =: c.$$

Note that for $\gamma=\rho=-1$ *we get a finite limit* $\overline{A}(t)/A(t) \underset{t\to\infty}{\longrightarrow} -1$ *and different from* $\gamma/(\gamma+\rho)=1/2$.

Let us now compute for $0<\gamma<-\rho$,

$$U(t)-\frac{a(t)}{\gamma} = \begin{cases} \frac{t^\gamma}{\gamma}\big(-t^{-\gamma}+o(t^{-\gamma})\big) & \text{if } 0<\gamma<1 \\ t\left(-\frac{3}{2t}+o\left(t^{-1}\right)\right) & \text{if } \gamma=1 \end{cases}$$

$$\underset{t\to\infty}{\longrightarrow} \begin{cases} -\frac{1}{\gamma} & \text{if } 0<\gamma<1 \\ -\frac{3}{2} & \text{if } \gamma=1 \end{cases} =: l.$$

in agreement with Theorem 2.1. Note however that $l\neq 0$ *for all* $0<\gamma<-\rho$ *and* $c=\pm\infty$ *for all region* $0<\gamma<-\rho$

On another side, for heavy tails, i.e., for $\gamma>0$,

$$\frac{\ln U(tx)-\ln U(t)-\gamma\ln x}{\widetilde{A}(t)} \underset{t\to\infty}{\longrightarrow} \frac{x^{\tilde\rho}-1}{\tilde\rho}, \quad \tilde\rho = \begin{cases} -\gamma & \text{if } 0<\gamma\le 1 \\ \rho=-1 & \text{if } \gamma>1 \end{cases},$$

$$\widetilde{A}(t) = \begin{cases} \gamma\ t^{-\gamma} & \text{if } 0<\gamma<1 \\ \frac{3}{2\ t} & \text{if } \gamma=1 \\ \frac{\gamma}{2t} & \text{if } \gamma>1 \end{cases} = \overline{A}(t)(1+o(1)),$$

now in agreement with the results in Corollary 3.1.

Example 4.4. *The most common heavy-tailed models with* $\widetilde{\rho}=-\gamma$ *and* $0<\gamma<-\rho$ *(then necessarily with* $l\neq 0$*), are such that*

$$U(t) \;=\; C\ t^\gamma\left(1+A\ t^{-\gamma}+B\ t^{-2\gamma}+o\left(t^{-2\gamma}\right)\right), \quad A,\ B\neq 0,\ C>0.$$

For these models,

$$U(tx)-U(t) = C\ \gamma\ t^\gamma\left(\frac{x^\gamma-1}{\gamma} - B\ t^{-2\gamma}\left(\frac{x^{-\gamma}-1}{-\gamma}\right)+o\left(t^{-2\gamma}\right)\right),$$

and

$$\frac{\frac{U(tx)-U(t)}{C\ \gamma\ t^{\gamma}}-\frac{x^{\gamma}-1}{\gamma}}{-B\ t^{-2\gamma}} \underset{t\to\infty}{\longrightarrow} \frac{x^{-\gamma}-1}{-\gamma},$$

i.e., $\rho+\gamma=-\gamma$*, or equivalently,* $\rho=-2\gamma$*. Then,* (2.5) *holds, provided that we choose*

$$a(t)=\frac{C\ \gamma\ t^{\gamma}}{1+B\ t^{-2\gamma}}, \qquad A(t)=2\ B\ \gamma\ t^{-2\gamma}$$

and

$$\frac{a(t)}{U(t)}=\gamma\left(1-A\ t^{-\gamma}-\left(2\ B-A^2\right)\ t^{-2\gamma}+o\left(t^{-2\gamma}\right)\right).$$

Consequently, with $\overline{A}(t)$*,* l *and* c *provided in* (2.8)*,* (2.9) *and* (2.10)*, respectively,*

$$\overline{A}(t)=-A\ \gamma\ t^{-\gamma}\left(1+O\left(t^{-\gamma}\right)\right), \quad \frac{\overline{A}(t)}{A(t)}=-\frac{A}{2\ B\ t^{-\gamma}}\left(1+O(t^{-\gamma})\right)\underset{t\to\infty}{\rightarrow} \pm\infty,$$

i.e., $c=\pm\infty$ *and*

$$U(t)-\frac{a(t)}{\gamma} \quad = \quad C\ t^{\gamma}\left(A\ t^{-\gamma}+2\ B\ t^{-2\gamma}+o(t^{-2\gamma})\right) \underset{t\to\infty}{\rightarrow} AC\neq 0,$$

i.e., $l=AC\neq 0$*, as mentioned at the very beginning of this example. Indeed, we could also have written*

$$U(t)=l+C\ t^{\gamma}\left(1+B\ t^{-2\gamma}+o\left(t^{-2\gamma}\right)\right), \ \text{as } t\to\infty.$$

5 The second order condition for a general tail, heavy tails and a third order framework

Note that for heavy-tailed models, the second order condition (2.5) implies a third order behaviour of the function $\ln U(t)$, whenever we are in the region $0<\gamma\le-\rho$, and $l\neq 0$, a region where $A(t)=o\big(\overline{A}(t)\big)$. Also, since $|\overline{A}|\in RV_{-\gamma}$, $|A|\in RV_{\rho}$ and $\overline{A}^2\in RV_{-2\gamma}$, then A dominates $\overline{A}^2$ if $\rho>-2\gamma$, but $\overline{A}^2$ dominates A if $\rho<-2\gamma$. From the Proof of Theorem 3.1, Case (i), the third order behaviour of $\ln U(t)$ may be written as

$$\lim_{t\to\infty}\frac{\frac{\ln U(tx)-\ln U(t)-\gamma\ln x}{\overline{A}(t)}-\frac{x^{-\gamma}-1}{-\gamma}}{\widetilde{B}(t)}=H_{\widetilde{\rho},\,\widetilde{\eta}}(x),$$

where H is defined in (2.5),

$$\widetilde{B}(t):=\begin{cases}-\overline{A}(t) & \text{if} \quad 0<\gamma<-\frac{\rho}{2}\\ \gamma\frac{A(t)}{\overline{A}(t)} & \text{if} \quad -\frac{\rho}{2}<\gamma<-\rho\end{cases}$$

and the second and third order parameters $\widetilde{\rho}$ and $\widetilde{\eta}$, respectively, are given by

$$\widetilde{\rho} = -\gamma, \qquad \widetilde{\eta} = \begin{cases} -\gamma & \text{if } \; 0 < \gamma < -\frac{\rho}{2} \\ \gamma + \rho & \text{if } \; -\frac{\rho}{2} < \gamma < -\rho \end{cases}.$$

Note that in the region $-\rho/2 < \gamma < -\rho$ we get $\widetilde{\rho} \neq \widetilde{\eta}$.

Remark 5.1. *For the case $\gamma = -\rho/2$, excluded from this note, everything depends on the relative behaviour of A and $\overline{A}^2$, both regularly varying functions with the same index of regular variation ρ.*

Note also that the situation $\widetilde{\eta} = \widetilde{\rho}$ is the one that most often happens in practice, for standard heavy-tailed models like Fréchet, Burr, the Generalized Pareto and Student's t d.f.'s. For these d.f.'s, (2.5) holds with $\rho = -2\gamma$. However, if we induce a shift $l \neq 0$ in the above mentioned models, this relation between γ and ρ no longer exists and we may cover all region $0 < \gamma < -\rho$.

Finally, we mention that for the extreme value model with d.f. EV_γ, we get

$$\rho = -1, \quad \widetilde{\rho} = -\gamma \quad \text{and} \quad \widetilde{\eta} = \begin{cases} -\gamma & \text{if } \; 0 \le \gamma \le 1/2 \\ \gamma - 1 & \text{if } \; 1/2 < \gamma < 1 \end{cases}.$$

For more details on the way the third order framework may be used in Statistics of Extremes, see Gomes, de Haan and Peng (2002) and Fraga Alves, Gomes and de Haan (2003), papers dealing with the estimation of the second order parameter ρ, and Gomes, Caeiro and Figueiredo (2004), a paper dealing with reduced bias extreme value index estimation.

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