The asymptotic joint locations of the largest and smallest extremes of a stationary sequence

Pereira, Luísa
University of Beira Interior, Department of Mathematics
Rua Marquês d´Ávila e Bolama, 6200-001 Covilhã, Portugal
E-mail: lpereira@mat.ubi.pt

Abstract

Under appropriate dependence conditions, we obtain the asymptotic independence of the joint locations of the largest extremes and the joint locations of the smallest extremes of a stationary sequence $\{X_n\}_{n\geq 1}$. The result obtained allows us to censure a sample, by ensuring that the set of observations that we selected contains the k largest and r smallest order statistics of the stationary sequence $\{X_n\}_{n\geq 1}$, with a pre-determined probability. We present an example of a 2-dependent sequence for which we can apply this result.

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1. Introduction

For a sequence of random variables $\mathbf{X} = \{X_n\}_{n\geq 1}$ and a family of real levels $\{u_n\}_{n\geq 1}$ we will denote the kth upper and the rth lower order statistics among $X_i, i \in I$, with $I \subset R_n = \{1, 2, \ldots, n\}$, respectively by $\overline{M}_n^{(k)}(I)$ and $\underline{M}_n^{(r)}(I)$ or simply by $\overline{M}_n^{(k)}$ and $\underline{M}_n^{(r)}$, when $I = R_n$.

We define the locations of $\overline{M}_n^{(k)}$ and $\underline{M}_n^{(r)}$, denoted respectively by $\overline{L}_n^{(k)}$ and $\underline{L}_n^{(r)}$, by

$$\overline{L}_n^{(k)} = min\left\{1 \leq j \leq n : \overline{M}_n^{(k)} = X_j\right\}$$

and

$$\underline{L}_n^{(r)} = min\left\{1 \leq j \leq n : \underline{M}_n^{(r)} = X_j\right\}.$$

Under appropriate dependence conditions Pereira *et al.* (2002) obtained the asymptotic independence of the locations of maximum and minimum of a stationary sequence $\mathbf{X}$.

Our aim, in this paper, is to obtain the asymptotic independence of the joint locations of the k largest order statistics and the joint locations of the s smallest order statistics.

The result obtained allows us to censure a sample, by ensuring that the set of observations that we selected contains the k largest and r smallest order statistics of the stationary sequence $\mathbf{X}$, with a pre-determined probability.

The rest of the paper is organized as follows: in section 2 we prove that under appropriate dependence conditions we have the asymptotic independence of an upper and a lower order statistics of a stationary sequence $\mathbf{X}$, which will be used in section 3. The main result in this section generalizes Davis´s result (1984) since it allows the presence of clusters of high values and clusters of low values in $\mathbf{X}$.

In section 3, we deal with the asymptotic distribution of the joint locations of the k largest and r smallest order statistics, as $n \longrightarrow \infty$. We prove under appropriate dependence conditions that for each $\varepsilon_1, \varepsilon_2 \in [0,1]$, $\left(\overline{M}_n^{(1)}(R_n \setminus ([1, n\varepsilon_1] \cap \mathbb{N})), \overline{M}_n^{(k)}([1, n\varepsilon_1] \cap \mathbb{N})\right)$ and $\left(\underline{M}_n^{(1)}(R_n \setminus ([1, n\varepsilon_2] \cap \mathbb{N})), \underline{M}_n^{(r)}([1, n\varepsilon_2] \cap \mathbb{N})\right)$

under linear normalization, are asymptotically independent, which, jointly with the asymptotic independence of a lower and an upper order statistics, will lead to the asymptotic independence of the joint locations of the k upper order statistics and the joint locations of the s lower order statistics.

2. Asymptotic independence of a lower and an upper extremes

In Davis (1984) it was proved that if a stationary sequence $\mathbf{X}$ satisfies appropriate long range and local dependence conditions and the sequences $\mathbf{X}$ and $-\mathbf{X} = \{-X_n\}_{n\geq 1}$ have extremal indexes equal to one, then we have the asymptotic independence of a lower and an upper extremes.

In this section we generalize this result of Davis since we allow the presence of clusters of high values and clusters of low values in $\mathbf{X}$.

The type of long range and local dependence conditions suitable in the present setting is a slight generalization of condition $D(\mathbf{X}, u_n, v_n)$ (Davis, 1982, 1983, 1984) and the $C(\mathbf{X}, u_n, v_n)$ condition (Davis, 1982, 1983, 1984).

Definition 2.1.: *Let $\mathbf{X}$ be a stationary sequence of random variables and $\{u_n\}_{n\geq 1}$, $\{v_n\}_{n\geq 1}$ sequences of real numbers. For each $1 \leq i \leq j$, set $B_i^j(u_n, v_n)$ as the $\sigma-$algebra generated by the events $\{v_n < X_s \leq u_n\}$, $i \leq s \leq j$, and, for $1 \leq l \leq n-1$*

$$\alpha^*_{n,l} = \max_{1\leq k\leq n-l} \left\{ |P(A\cap B) - P(A)P(B)| : A \in \mathcal{B}_1^k(u_n, v_n), B \in \mathcal{B}_{k+l}^n(u_n, v_n) \right\}. \tag{2.1}$$

The condition $\Delta^*(\mathbf{X}, u_n, v_n)$ is said to hold if there exists a sequence $l_n = o(n)$, as $n \longrightarrow \infty$, such that

$$\alpha^*_{n,l_n} \xrightarrow[n\to\infty]{} 0. \tag{2.2}$$

By assuming (2.2), when we take in (2.1) only the events $A = \bigcap_{s=1}^{p} \{v_n < X_{i_s} \leq u_n\}$ and $B = \bigcap_{s=1}^{q} \{v_n < X_{j_s} \leq u_n\}$, jointly to the conditions $D(u_n)$ for $\{X_n\}_{n\geq 1}$ and $D(-v_n)$ for $\{-X_n\}_{n\geq 1}$ (Leadbetter, 1988) we obtain the condition $D(\mathbf{X}, u_n, v_n)$ of Davis.

The condition $C(\mathbf{X}, u_n, v_n)$ holds if

$$\limsup_{n\longrightarrow\infty} \sum_{j=1}^{r_n-1} \left(P(X_1 > u_n, X_{j+1} \leq v_n) + P(X_1 \leq v_n, X_{j+1} > u_n)\right) = 0,$$

where $r_n = \left[\frac{n}{k_n}\right]$, $[s]$ denotes the greatest integer not greater than s and $\{k_n\}_{n\geq 1}$ is a sequence of integer numbers such that $k_n \longrightarrow \infty$.

Under $C(\mathbf{X}, u_n, v_n)$ and $D(\mathbf{X}, u_n, v_n)$ conditions, Davis (1982) proves the asymptotic independence of the maximum and minimum.

Proposition 2.1.: *Let $\mathbf{X}$ be a stationary sequence and θ_1 and θ_2 the extremal indexes of $\mathbf{X}$ and $-\mathbf{X}$, respectively. Let $\{a_n > 0\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$, $\{c_n > 0\}_{n\geq 1}$ and $\{d_n\}_{n\geq 1}$ be sequences of constants such that*

$$P\left(a_n^{-1}\left(\overline{M}_n^{(1)} - b_n\right) \leq x\right) \xrightarrow[n\to\infty]{} G^{\theta_1}(x) \tag{2.3}$$

and

$$P\left(c_n^{-1}\left(\underline{M}_n^{(1)} - d_n\right) \leq y\right) \xrightarrow[n\to\infty]{} 1 - (1 - H(y))^{\theta_2} \tag{2.4}$$

where $G(x)$ and $H(-y)$ are extreme value distributions. If, for each $x, y \in \mathbb{R}$, and $u_n = u_n(x) = a_n x + b_n$, $v_n = v_n(y) = c_n y + d_n$, the conditions $D(\mathbf{X}, u_n, v_n)$ and $C(\mathbf{X}, u_n, v_n)$ hold with $\{k_n\}_{n\geq 1}$ and $\{l_n\}_{n\geq 1}$ sequences of integer numbers verifying

$$(2.5) \qquad k_n \xrightarrow[\mathbf{n}\to\infty]{} \infty, \ \frac{k_n l_n}{n} \xrightarrow[\mathbf{n}\to\infty]{} 0, \ k_n \alpha_{n,l_n} \xrightarrow[n\to\infty]{} 0,$$

where α_{n,l_n} is the mixing coefficient of the $D(\mathbf{X}, u_n, v_n)$ condition, then

$$P(a_n^{-1}\left(\overline{M}_n^{(1)} - b_n\right) \leq x, c_n^{-1}\left(\underline{M}_n^{(1)} - d_n\right) \leq y) \xrightarrow[n\to\infty]{} G^{\theta_1}(x)\left(1 - (1 - H(y_1))^{\theta_2}\right).$$

The joint limiting distribution of $\left(\frac{\overline{M}_n^{(k)} - b_n}{a_n}, \frac{\underline{M}_n^{(r)} - d_n}{c_n}\right)$, with $a_n > 0, c_n > 0$, b_n and d_n real constants, will be discussed in terms of the convergence in distribution of the sequence of bidimensional point processes $\{(S_n[\mathbf{X}, u_n], S_n[-\mathbf{X}, -v_n])\}_{n\geq 1}$, where $u_n = u_n(x) = a_n x + b_n$, $v_n = v_n(y) = c_n y + d_n$ and the point process of exceedances of w_n, $n \geq 1$, by $\mathbf{Y} = \{Y_n\}_{n\geq 1}$, is defined on $[0,1]$, by

$$S_n[\mathbf{Y}, w_n](\cdot) = \sum_{i=1}^{n} \mathbb{I}_{\{Y_i > w_n\}} \delta_{\frac{i}{n}}(\cdot),$$

with δ_a the Dirac measure at $a \in \mathbb{R}$.

Under the condition $\Delta^*(\mathbf{X}, u_n, v_n)$ it is possible to characterize the distributional limits for $\{(S_n[\mathbf{X}, u_n], S_n[-\mathbf{X}, -v_n])\}_{n\geq 1}$ and to set a necessary and sufficient condition for its convergence in distribution. We present that result, which is a corollary of the main result of Nandagopalan (1990) on the bidimensional sequence of point processes of exceedances applied to the sequences $\{(X_n, -X_n)\}_{n\geq 1}$ and $\{(u_n, -v_n)\}_{n\geq 1}$.

Proposition 2.2.: *Let $\mathbf{X}$ be a stationary sequence. Suppose that $\Delta^*(\mathbf{X}, u_n, v_n)$ holds and $\{(S_n[\mathbf{X}, u_n], S_n[-\mathbf{X}, -v_n])\}_{n\geq 1}$ converges in distribution to some point process $\mathbf{S}$. Then $\mathbf{S}$ is necessarily a compound Poisson process with Laplace transform*

$$L_{\mathbf{S}}(f_1, f_2) = exp\left(-\nu \int_{[0,1]} \int_{\mathbb{N}_0^2 \setminus \{\mathbf{0}\}} \left(1 - exp\left(-\sum_{j=1}^{2} y_j f_j(x)\right)\right) d\Pi(y_1, y_2)\, dx\right)$$

for each non-negative measurable functions f_1 and f_2 on $[0,1]$, where

$$(2.6) \qquad \nu = -log \lim_{n\to\infty} P\left(S_n[X_n, u_n]([0,1]) = 0, S_n[-X_n, -v_n]([0,1]) = 0\right)$$

and

$$(2.7) \qquad \Pi(y_1, y_2) = \lim_{n\to\infty} \Pi_n(y_1, y_2), \quad (y_1, y_2) \in \mathbb{N}_0^2 \setminus \{\mathbf{0}\}$$

with

$$\Pi_n(y_1, y_2) = P\left(\sum_{i=1}^{r_n} \mathbb{I}_{\{X_i > u_n\}} = y_1, \sum_{i=1}^{r_n} \mathbb{I}_{\{X_i \leq v_n\}} = y_2 \mid \sum_{i=1}^{r_n} \mathbb{I}_{\{X_i > u_n\}} + \sum_{i=1}^{r_n} \mathbb{I}_{\{X_i \leq v_n\}} > 0\right)$$

$r_n = \left[\frac{n}{k_n}\right]$ *and $\{k_n\}_{n\geq 1}$ is a sequence of integer numbers such that*

$$(2.8) \qquad \frac{k_n l_n}{n} \xrightarrow[n\to\infty]{} 0, \quad k_n \alpha^*_{n,l_n} \xrightarrow[n\to\infty]{} 0, \ k_n \xrightarrow[n\to\infty]{} \infty.$$

Moreover, if (2.6) and (2.7) hold for some sequence $\{k_n\}_{n\geq 1}$ *satisfying (2.8), then the sequence* $\{(S_n[\mathbf{X}, u_n], S_n[-\mathbf{X}, -v_n])\}_{n\geq 1}$ *converges in distribution to the above compound Poisson process.*

By considering in Proposition 2.2. only a sequence $\{X_n\}_{n\geq 1}$ of random variables and a sequence of real numbers $\{u_n\}_{n\geq 1}$, we obtain the result of Hsing *et al.* (1988).

As a consequence of Proposition 2.2. and its unidimensional version applied to the sequences $\{S_n[\mathbf{X}, u_n]\}_{n\geq 1}$ and $\{S_n[-\mathbf{X}, -v_n]\}_{n\geq 1}$, we conclude that if $\mathbf{X}$ and $-\mathbf{X}$ have extremal indexes θ_1 and θ_2, there exists sequences $\{a_n > 0\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$, $\{c_n > 0\}_{n\geq 1}$ and $\{d_n\}_{n\geq 1}$ satisfying (2.3) and (2.4), for each $x, y \in \mathbb{R}$ and $u_n = u_n(x) = a_n x + b_n$, $v_n = v_n(y) = c_n y + d_n$, $\Delta^*(\mathbf{X}, u_n, v_n)$ holds and $\{S_n[\mathbf{X}, u_n]\}_{n\geq 1}$, $\{S_n[-\mathbf{X}, -v_n]\}_{n\geq 1}$ and $\{(S_n[\mathbf{X}, u_n], S_n[-\mathbf{X}, -v_n])\}_{n\geq 1}$ converge, then

$$\begin{aligned}
G_k(x) &= \lim_{n\to\infty} P\left(\overline{M}_n^{(k)} \leq u_n(x)\right) = G^{\theta_1}(x)\left(1 + \sum_{i=1}^{k-1}\sum_{j=1}^{i} \frac{\left(-logG^{\theta_1}(x)\right)^j}{j!}\Pi_1^{*^j}(i)\right),\\
H_r(y) &= \lim_{n\to\infty} P\left(\underline{M}_n^{(r)} \leq v_n(y)\right) = 1 - (1 - H(y))^{\theta_2}\left(1 + \sum_{i=1}^{r-1}\sum_{j=1}^{i} \frac{\left(-log(1 - H(y))^{\theta_2}\right)^j}{j!}\Pi_2^{*^j}(i)\right),\\
F_{k,r}(x, y) &= \lim_{n\to\infty} P\left(\overline{M}_n^{(k)} \leq u_n(x), \underline{M}_n^{(r)} \leq v_n(y)\right)\\
&= G_k(x) - (G_1(x) - F_{1,1}(x, y))\left[1 + \sum_{i_1=0}^{k-1}\sum_{\substack{i_2=0\\ i_1+i_2>0}}^{r-1}\sum_{j=1}^{max(i_1,i_2)} \frac{(-log(G_1(x) - F_{1,1}(x, y)))^j}{j!}\Pi^{*^j}(i_1, i_2)\right]
\end{aligned}$$

where, for each $x, y \in \mathbb{N}$,

$$\begin{aligned}
\Pi_1(x) &= \lim_{n\to\infty} \Pi_{n,1}(x) = \lim_{n\to\infty} P\left(\sum_{i=1}^{r_n} \mathbb{I}_{\{X_i > u_n\}} = x \mid \sum_{i=1}^{r_n} \mathbb{I}_{\{X_i > u_n\}} > 0\right),\\
\Pi_2(y) &= \lim_{n\to\infty} \Pi_{n,2}(y) = \lim_{n\to\infty} P\left(\sum_{i=1}^{r_n} \mathbb{I}_{\{X_i \leq v_n\}} = y \mid \sum_{i=1}^{r_n} \mathbb{I}_{\{X_i \leq v_n\}} > 0\right),
\end{aligned}$$

and $\Pi(x, y)$ is defined as previously.

By attending that the condition $C(\mathbf{X}, u_n, v_n)$ allows to despise, for n large, the probability that happening in to an interval of length r_n "brusque ascent" and "brusque descent" the next result establishes that if the limit distribution of Π_n exists then it will be concentrated on

$$\left\{(z_1, z_2) \in \mathbb{N}_{\mathbf{0}}^2 \setminus \{\mathbf{0}\} : z_1 = 0 \vee z_2 = 0\right\}.$$

Proposition 2.3.: *Let* $\{u_n\}_{n\geq 1}$ *and* $\{v_n\}_{n\geq 1}$ *be sequences of real numbers such that*

$$nP(X_1 > u_n) \xrightarrow[n\to\infty]{} \tau_1 > 0 \quad and \quad nP(X_1 < v_n) \xrightarrow[n\to\infty]{} \tau_2 > 0. \tag{2.9}$$

If $\mathbf{X}$ *and* $-\mathbf{X}$ *have extremal indexes* θ_1 *and* θ_2*, respectively,* $\mathbf{X}$ *is a stationary sequence and the conditions* $D(\mathbf{X}, u_n, v_n)$ *and* $C(\mathbf{X}, u_n, v_n)$ *hold with* $\{k_n\}_{n\geq 1}, \{l_n\}_{n\geq 1}, \{u_n\}_{n\geq 1}$ *and* $\{v_n\}_{n\geq 1}$ *verifying (2.5), then for each* $(y_1, y_2) \in \mathbb{N}_{\mathbf{0}}^2 \setminus \{\mathbf{0}\}$ *we have*

(i) $\lim_{n\to\infty} \Pi_n(y_1, y_2) = 0 \quad if \quad y_1 y_2 \neq 0;$

(ii) $\lim_{n\to\infty} \left(\Pi_n(y, 0) - \frac{\nu_1}{\nu_1+\nu_2}\Pi_{n,1}(y)\right) = \lim_{n\to\infty} \left(\Pi_n(0, y) - \frac{\nu_2}{\nu_1+\nu_2}\Pi_{n,2}(y)\right) = 0, \quad y \in \mathbb{N}.$

Proof: For sake of simplicity, we denote $\sum_{i=1}^{r_n} \mathbb{I}_{\{X_i > u_n\}}$ and $\sum_{i=1}^{r_n} \mathbb{I}_{\{X_i < v_n\}}$, respectively, by $S_{r_n}^{(1)}$ and $S_{r_n}^{(2)}$. By the Lemma of asymptotic independence of maxima over disjoint intervals and applying the

Proposition 2.1., it follows that

$$\begin{aligned}
&\lim_{n\to\infty} k_n \left(1 - P\left(S_{r_n}^{(1)} = 0, S_{r_n}^{(2)} = 0\right)\right) \\
= &\lim_{n\to\infty} \left(-logP^{k_n}\left(S_{r_n}^{(1)} = 0, S_{r_n}^{(2)} = 0\right)\right) \\
= &-\log \lim_{n\to\infty} P\left(\overline{M}_n^{(1)} \le u_n, \underline{M}_n^{(1)} > u_n\right) = \nu_1 + \nu_2 = \theta_1\tau_1 + \theta_2\tau_2.
\end{aligned}$$

Then, for all $(y_1, y_2) \in \mathbb{N}_{\mathbf{0}}^2 \setminus \{\mathbf{0}\}$ such that $y_1 y_2 \neq 0$, we have

$$\Pi_n(y_1, y_2) = \frac{k_n P\left(S_{r_n}^{(1)} = y_1, S_{r_n}^{(2)} = y_2\right)}{k_n P\left(S_{r_n}^{(1)} + S_{r_n}^{(2)} > 0\right)} = \frac{k_n P\left(S_{r_n}^{(1)} = y_1, S_{r_n}^{(2)} = y_2\right)}{\nu_1 + \nu_2}(1 + o(1)). \tag{2.10}$$

Since $\mathbf{X}$ is a stationary sequence, $C(\mathbf{X}, u_n, v_n)$ holds and $k_n P(X_1 < v_n) + k_n P(X_1 > u_n) = o(1)$, we obtain

$$\begin{aligned}
&k_n P\left(S_{r_n}^{(1)} \neq 0, S_{r_n}^{(2)} \neq 0\right) \\
= &k_n P\left(S_{r_n}^{(1)} \neq 0, S_{r_n}^{(2)} \neq 0, v_n \le X_1 \le u_n\right) + o(1) \\
\le &k_n P\left(\bigcup_{1\le i<j\le n} \{X_i > u_n, X_j < v_n\}\right) + o(1) \\
\le &k_n r_n \sum_{j=2}^{r_n} P(X_1 > u_n, X_j < v_n) + P(X_1 < v_n, X_j > u_n) = o(1),
\end{aligned}$$

which, with (2.10) allow us to conclude that $\Pi_n(y_1, y_2) = o(1)$ if $y_1 y_2 \neq 0$.

(ii) Taking $y_2 = 0$ in (2.10), we have

$$\begin{aligned}
\Pi_n(y, 0) &= \frac{k_n P\left(S_{r_n}^{(1)} = y, S_{r_n}^{(2)} = 0\right)}{\nu_1 + \nu_2}(1 + o(1)) \\
&= \frac{k_n P\left(S_{r_n}^{(1)} = y, S_{r_n}^{(2)} = 0, v_n \le X_1 \le u_n\right) + o(1)}{\nu_1 + \nu_2}(1 + o(1)) \\
&= \frac{k_n P\left(S_{r_n}^{(1)} = y\right) + o(1)}{\nu_1 + \nu_2}(1 + o(1)),
\end{aligned}$$

since

$$k_n P\left(S_{r_n}^{(1)} = y, S_{r_n}^{(2)} \neq 0, v_n \le X_1 \le u_n\right) = o(1).$$

So,

$$\begin{aligned}
\Pi_n(y, 0) &= \frac{k_n \Pi_{n,1}(y) P\left(S_{r_n}^{(1)} > 0\right) + o(1)}{\nu_1 + \nu_2}(1 + o(1)) \\
&= \Pi_{n,1}(y) \frac{\nu_1 + o(1)}{\nu_1 + \nu_2}(1 + o(1)).
\end{aligned}$$

The second convergence in (ii) is proved by using analogous arguments.

We are now ready to prove the asymptotic independence of an upper and a lower order statistics.

Corollary 2.1.: *Let* $\mathbf{X}$ *be a stationary sequence and suppose that* $C(\mathbf{X}, u_n, v_n)$ *and* $\Delta^*(\mathbf{X}, u_n, v_n)$ *hold with* $\{k_n\}_{n\ge1}$ *and* $\{l_n\}_{n\ge1}$ *sequences of integers numbers verifying (2.8). If the sequences* $\mathbf{X}$ *and*

$-\mathbf{X}$ *have extremal indexes* θ_1 *and* θ_2*, respectively, satisfy (2.9) and, for each* $i=1,2$, $\Pi_{n,i}$ *converges weakly to a distribution* Π_i*, then* $\{(S_n[\mathbf{X},u_n], S_n[-\mathbf{X},-v_n])\}_{n\geq 1}$ *converges in distribution to the point process with Laplace transform*

$$L_{S[\nu_1,\Pi_1]}(f_1)L_{S[\nu_2,\Pi_2]}(f_2),$$

for any f_1, f_2 *non-negative measurable functions on* $[0,1]$.

Proof: Since the conditions of Proposition 2.2. are satisfied with

$$\nu=\nu_1+\nu_2=\theta_1\tau_1+\theta_2\tau_2$$

and

$$\Pi(y_1,y_2)=\begin{cases}0 & \text{if } y_1y_2\neq 0\\ \frac{\nu_1}{\nu_1+\nu_2}\Pi_1(y_1) & \text{if } y_1\neq 0\wedge y_2=0\\ \frac{\nu_2}{\nu_1+\nu_2}\Pi_2(y_2) & \text{if } y_1=0\wedge y_2\neq 0\end{cases}$$

then the limit point process is a bidimensional compound Poisson process with intensity $\nu_1+\nu_2$ and distribution of multiplicities Π, with Laplace transform

$$\begin{aligned}
& L_{\mathbf{S}[\nu_1+\nu_2,\Pi]}(f_1,f_2)\\
= & \ exp\left(-(\nu_1+\nu_2)\int_{[0,1]}\int_{\mathbb{N}_{\mathbf{0}}^2\setminus\{\mathbf{0}\}}\left(1-exp\left(-\sum_{j=1}^{2}y_jf_j(x)\right)\right)d\Pi(y_1,y_2)\,dx\right)\\
= & \ exp\left(-(\nu_1+\nu_2)\int_{[0,1]}\left(\int_{\mathbb{N}}(1-exp(-y_1f_1(x)))\frac{\nu_1}{\nu_1+\nu_2}d\Pi_1(y_1)dx\right.\right.\\
& \left.\left.+\int_{\mathbb{N}}(1-exp(-y_2f_2(x)))\frac{\nu_2}{\nu_1+\nu_2}d\Pi_2(y_2)dx\right)\right)\\
= & \ L_{S[\nu_1,\Pi_1]}(f_1)\times L_{S[\nu_2,\Pi_2]}(f_1).
\end{aligned}$$

3. Asymptotic independence of the joint locations of the k largest extremes and the joint locations of the r smallest extremes

In Ferreira *et al.* *(2002)* was introduced a slight generalization of $\Delta(u_n)-$condition which enable us to deal with exceedances of multiple levels and their locations in disjoint intervals.

Definition 3.1.: *Let* $\mathbf{X}$ *be a stationary sequence of random variables and* $\left\{u_n^{(i)}\right\}_{n\geq 1}$, $i=1,2,$ *sequences of real numbers. For each* $1\leq r\leq s$*, set* $B_r^s(u_n^{(i)^*})$ *as the* $\sigma-$*algebra generated by the events* $\left\{X_t\leq u_n^{(i)^*}\right\}$, $r\leq t\leq s$*, where* $u_n^{(i)^*}\in\left\{u_n^{(1)},u_n^{(2)}\right\}$*, and, for* $1\leq l\leq n-1$

$$\alpha_{n,l}^{(2)}=\max_{1\leq k\leq n-l}\left\{|P(A\cap B)-P(A)P(B)|:A\in\mathcal{B}_1^k(u_n^{(1)}),B\in\mathcal{B}_{k+l}^n(u_n^{(2)})\right\}.$$

The condition $\Delta_2\left(\mathbf{X},u_n^{(1)},u_n^{(2)}\right)$ *is said to hold if there exists a sequence* $l_n=o(n)$*, as* $n\longrightarrow\infty$*, such that* $\alpha_{n,l_n}^{(2)}\underset{n\to\infty}{\longrightarrow}0.$

Under $\Delta_2\left(\mathbf{X},u_n^{(1)},u_n^{(2)}\right)$, for all $u_n^{(1)}=u_n(x_1)=a_nx_1+b_n, u_n^{(2)}=u_n(x_2)=a_nx_2+b_n, x_1,x_2\in\mathbb{R}$, and by supposing that the sequence $\left\{S_n\left[\mathbf{X},u_n^{(i)^*}\right]\right\}$, with $u_n^{(i)^*}\in\left\{u_n^{(1)},u_n^{(2)}\right\}$, converges, Ferreira *et al.* (2002) obtained the asymptotic behavior of the joint locations of the k largest order statistics:

$\forall\varepsilon \in [0,1]$,

$$\underset{n\to\infty}{lim} P\left(\bigcap_{i=1}^{k}\left\{\overline{L}_n^{(i)} \le n\varepsilon\right\}\right) = \varepsilon - \sum_{s=1}^{k-1}\varepsilon^s(1-\varepsilon)\sum_{t=s}^{k-1}\Pi_1^{*^s}(t),$$

where $\Pi_1^{*^s}$ is the *sth* convolution of the probability distribution Π_1 on $\mathbb{N}$, defined in the previous section.

Since the *rth* lower extreme among X_i, $i \le n$, $\underline{M}_n^{(r)}$, is simply given as the *rth* upper extreme among $-X_i$, $i \le n$, using analogous arguments as in Ferreira *et al.* (2002), we conclude that if $\Delta_2\left(-\mathbf{X}, -v_n^{(1)}, -v_n^{(2)}\right)$ holds for all $v_n^{(1)} = v_n(x_1) = c_n x_1 + d_n, v_n^{(2)} = v_n(x_2) = c_n x_2 + d_n, x_1, x_2 \in \mathbb{R}$, and $\left\{S_n\left[-\mathbf{X}, -v_n^{(i)^*}\right]\right\}$, with $v_n^{(i)^*} \in \left\{v_n^{(1)}, v_n^{(2)}\right\}$, converges,then

$$\underset{n\to\infty}{lim} P\left(\bigcap_{i=1}^{r}\left\{\underline{L}_n^{(i)} \le n\varepsilon\right\}\right) = \varepsilon - \sum_{s=1}^{r-1}\varepsilon^s(1-\varepsilon)\sum_{j=s}^{r-1}\Pi_2^{*^s}(j).$$

In the following, we will define generalizations of the conditions defined in the previous section in order to obtain for every subsets I, J, I', J' of $[0,1]$ such that $I \cup J = I' \cup J' = [0,1]$ and $I \cap J = I' \cap J' = \emptyset$, the asymptotic independence of the events

$$\left\{\overline{M}_n^{(1)}(I) \le u_n(x_1), \overline{M}_n^{(k)}(J) \le u_n(x_2)\right\} \quad \text{and} \quad \left\{\underline{M}_n^{(1)}(I') > v_n(y_1), \underline{M}_n^{(r)}(J') > v_n(y_2)\right\},$$

where the levels $u_n(x_i), v_n(y_i), i = 1,2$, satisfy $n_1 n_2 P\left(X_1 > u_n(x_i)\right) \underset{n\to\infty}{\longrightarrow} \tau_i$, $n_1 n_2 P\left(X_1 \le v_n(y_i)\right) \underset{n\to\infty}{\longrightarrow} \tau_i'$. This in turn will lead to the asymptotic independence of the joint locations of the k upper order statistics and the joint locations of the r lower order statistics.

Definition 3.2.: *Let* $\mathbf{X}$ *be a stationary sequence of random variables and* $\left\{u_n^{(i)}\right\}_{n\ge1}$, $\left\{v_n^{(i)}\right\}_{n\ge1}$, $i = 1,2$, *sequences of real numbers. The condition* $\Delta_2^*(\mathbf{X},(u_n^{(1)}, u_n^{(2)}), (v_n^{(1)}, v_n^{(2)}))$ *holds if in (2.1) we consider the* $\sigma-$*algebra generated by the events* $\left\{v_n^{(i)^*} < X_s \le u_n^{(i)^*}\right\}$, *where* $u_n^{(i)^*} \in \left\{u_n^{(1)}, u_n^{(2)}\right\}$ *and* $v_n^{(i)^*} \in \left\{v_n^{(1)}, v_n^{(2)}\right\}$.

We shall define the mixing coefficient in condition $\Delta_2^*(\mathbf{X},(u_n^{(1)}, u_n^{(2)}), (v_n^{(1)}, v_n^{(2)}))$ by $\alpha_{n,l_n}^{*(2)}$.

Definition 3.3.: *Let* $\mathbf{X}$ *be a stationary sequence. The condition* $C(\mathbf{X},(u_n^{(1)}, u_n^{(2)}), (v_n^{(1)}, v_n^{(2)}))$ *holds if*

$$\underset{n\longrightarrow\infty}{limsup} k_n \sum_{i=1}^{r_n}\sum_{j=1}^{r_n} P(X_i > u_n^{(i)^*}, X_j \le v_n^{(i)^*}) = 0,$$

where $u_n^{(i)^*} \in \left\{u_n^{(1)}, u_n^{(2)}\right\}$, $v_n^{(i)^*} \in \left\{v_n^{(1)}, v_n^{(2)}\right\}$ *and* $\{k_n\}_{n\ge1}$ *is a sequence of integer numbers such that* $k_n \longrightarrow \infty$.

If $u_n^{(1)} = u_n^{(2)} = u_n$ and $v_n^{(1)} = v_n^{(2)} = v_n$ we obtain conditions $\Delta^*(\mathbf{X}, u_n, v_n)$ and $C(\mathbf{X}, u_n, v_n)$.

Lemma 3.1.: *Let* $\mathbf{X}$ *be a stationary sequence and suppose that, for each* $x_1, x_2, y_1, y_2 \in \mathbb{R}$, *and* $u_n^{(i)} = u_n(x_i) = a_n x_i + b_n$, $v_n^{(i)} = v_n(y_i) = c_n y_i + d_n$, $i = 1,2$, *the condition* $\Delta_2^*(\mathbf{X},(u_n^{(1)}, u_n^{(2)}), (v_n^{(1)}, v_n^{(2)}))$ *holds and the sequence* $\left\{\left(S_n\left[\mathbf{X}, u_n^{(i)^*}\right], S_n\left[-\mathbf{X}, -v_n^{(i)^*}\right]\right)\right\}_{n\ge1}$, *where* $u_n^{(i)^*} \in \left\{u_n^{(1)}, u_n^{(2)}\right\}$, $v_n^{(i)^*} \in \left\{v_n^{(1)}, v_n^{(2)}\right\}$, *converges. Then, for disjoint subsets* I *and* J *of* $[0,1]$, *we have the asymptotic independence of*

$$\left(S_n\left[\mathbf{X}, u_n^{(i)^*}\right](I), S_n\left[-\mathbf{X}, -v_n^{(i)^*}\right](I)\right) \text{ and } \left(S_n\left[\mathbf{X}, u_n^{(i)^*}\right](J), S_n\left[-\mathbf{X}, -v_n^{(i)^*}\right](J)\right).$$

Lemma 3.2.: *Let* $\mathbf{X}$ *and* $-\mathbf{X}$ *be sequences with extremal indexes* θ_1 *and* θ_2*, respectively. If* $C(\mathbf{X},(u_n^{(1)},u_n^{(2)}),(v_n^{(1)},v_n^{(2)}))$ *and* $\Delta_2^*(\mathbf{X},(u_n^{(1)},u_n^{(2)}),(v_n^{(1)},v_n^{(2)}))$ hold, *where the levels* $u_n^{(i)},v_n^{(i)}$*,* $i=1,2$*, satisfy*

$$nP(X_1>u_n^{(i)})\xrightarrow[n\to\infty]{}\tau_i>0 \quad and \quad nP(X_1\le v_n^{(i)})\xrightarrow[n\to\infty]{}\tau_i'>0,$$

the sequence $\{k_n\}_{n\ge1}$ *verifies*

$$\frac{k_nl_n}{n}\xrightarrow[n\to\infty]{}0,\quad k_n\alpha^{*(2)}_{n,l_n}\xrightarrow[n\to\infty]{}0,\quad k_n\xrightarrow[n\to\infty]{}\infty,$$

and, for each $u_n^{(i)^*}\in\left\{u_n^{(1)},u_n^{(2)}\right\}$*,* $v_n^{(i)^*}\in\left\{v_n^{(1)},v_n^{(2)}\right\}$*, the sequence* $\left\{\left(S_n\left[\mathbf{X},u_n^{(i)^*}\right],S_n\left[-\mathbf{X},-v_n^{(i)^*}\right]\right)\right\}_{n\ge1}$ *converges, then, by considering* $A_{\varepsilon_i}=\{1,\ldots,[n\varepsilon_i]\},i=1,2,$ *we have, for each* $\varepsilon_1,\varepsilon_2\in(0,1]$*,*

$$\begin{aligned}&P\left(\overline{M}_n^{(1)}(R_n\setminus A_{\varepsilon_1})\le u_n^{(1)},\overline{M}_n^{(k)}(A_{\varepsilon_1})\le u_n^{(2)},\underline{M}_n^{(1)}(R_n\setminus A_{\varepsilon_2})>v_n^{(1)},\underline{M}_n^{(r)}(A_{\varepsilon_2})>v_n^{(2)}\right)\\&-P\left(\overline{M}_n^{(1)}(R_n\setminus A_{\varepsilon_1})\le u_n^{(1)},\overline{M}_n^{(k)}(A_{\varepsilon_1})\le u_n^{(2)}\right)\times\\&\qquad P\left(\underline{M}_n^{(1)}(R_n\setminus A_{\varepsilon_2})>v_n^{(1)},\underline{M}_n^{(r)}(A_{\varepsilon_2})>v_n^{(2)}\right)\xrightarrow[n\to\infty]{}0.\end{aligned}$$

Proof: By supposing, for example, that $\epsilon_1<\epsilon_2$, let $I_1=(0,\epsilon_1]$, $I_2=(\varepsilon_1,\varepsilon_2]$ and $I_3=(\varepsilon_2,1]$. Then, by applying Lemma 3.1, we obtain

$$\begin{aligned}&\lim_{n\to\infty}P\left(\overline{M}_n^{(1)}(R_n\setminus A_{\varepsilon_1})\le u_n^{(1)},\overline{M}_n^{(k)}(A_{\varepsilon_1})\le u_n^{(2)},\underline{M}_n^{(1)}(R_n\setminus A_{\varepsilon_2})>v_n^{(1)},\underline{M}_n^{(r)}(A_{\varepsilon_2})>v_n^{(2)}\right)\\=&\lim_{n\to\infty}P\left(S_n\left[\mathbf{X},u_n^{(2)}\right](I_1)\le k-1,S_n\left[-\mathbf{X},-v_n^{(2)}\right](I_1)\le r-1\right)\times\\&P\left(S_n\left[\mathbf{X},u_n^{(1)}\right](I_2)=0,S_n\left[-\mathbf{X},-v_n^{(2)}\right](I_2)\le r-1\right)\times\\&P\left(S_n\left[\mathbf{X},u_n^{(1)}\right](I_3)=0,S_n\left[-\mathbf{X},-v_n^{(1)}\right](I_3)=0\right),\end{aligned}$$

and since $C(\mathbf{X},(u_n^{(1)},u_n^{(2)}),(v_n^{(1)},v_n^{(2)}))$ *and* $\Delta_2^*(\mathbf{X},(u_n^{(1)},u_n^{(2)}),(v_n^{(1)},v_n^{(2)}))$ imply $C(\mathbf{X},u_n^{(i)},v_n^{(j)})$ *and* $\Delta^*(\mathbf{X},u_n^{(i)},v_n^{(j)})$, for each $i,j\in\{1,2\}$, it follows, from Corollary 2.1. that

$$\begin{aligned}&\lim_{n\to\infty}P\left(\overline{M}_n^{(1)}(R_n\setminus A_{\varepsilon_1})\le u_n^{(1)},\overline{M}_n^{(k)}(A_{\varepsilon_1})\le u_n^{(2)},\underline{M}_n^{(1)}(R_n\setminus A_{\varepsilon_2})>v_n^{(1)},\underline{M}_n^{(r)}(A_{\varepsilon_2})>v_n^{(2)}\right)\\=&\lim_{n\to\infty}P\left(S_n\left[\mathbf{X},u_n^{(2)}\right](I_1)\le k-1\right)\times P\left(S_n\left[-\mathbf{X},-v_n^{(2)}\right](I_1)\le r-1\right)\times\\&P\left(S_n\left[\mathbf{X},u_n^{(1)}\right](I_2)=0\right)\times P\left(S_n\left[-\mathbf{X},-v_n^{(2)}\right](I_2)\le r-1\right)\times\\&P\left(S_n\left[\mathbf{X},u_n^{(1)}\right](I_3)=0\right)\times P\left(S_n\left[-\mathbf{X},-v_n^{(1)}\right](I_3)=0\right)\end{aligned}$$

Therefore, by using stationarity and attending that the sequences $\mathbf{X}$ and $-\mathbf{X}$ have extremal indexes θ_1 and θ_2, respectively, and $\Delta(\mathbf{X},u_n^{(i)}),\Delta(-\mathbf{X},-v_n^{(i)})$, $i=1,2$, hold, we obtain

$$\begin{aligned}&\lim_{n\to\infty}P\left(\overline{M}_n^{(1)}(R_n\setminus A_{\varepsilon_1})\le u_n^{(1)},\overline{M}_n^{(k)}(A_{\varepsilon_1})\le u_n^{(2)},\underline{M}_n^{(1)}(R_n\setminus A_{\varepsilon_2})>v_n^{(1)},\underline{M}_n^{(r)}(A_{\varepsilon_2})>v_n^{(2)}\right)\\=&e^{-\theta_2\tau_1'(1-\varepsilon_2)}\times e^{-\theta_1\tau_1(1-\varepsilon_1)}\times e^{-\theta_1\tau_2\varepsilon_1}\times\left[1+\sum_{i=1}^{k-1}\sum_{j=1}^{i}\frac{(\theta_1\tau_2\varepsilon_1)^j}{j!}\Pi_1^{*^j}(i)\right]\times\\&\left[1-e^{-\theta_2\tau_2'\varepsilon_2}\left(1+\sum_{i=1}^{r-1}\sum_{j=1}^{i}\frac{(\theta_2\tau_2'\varepsilon_2)^j}{j!}\Pi_2^{*^j}(i)\right)\right].\end{aligned}$$

Now, since $\Delta_2(\mathbf{X},u_n^{(1)},u_n^{(2)})$ and $\Delta_2(-\mathbf{X},-v_n^{(1)},-v_n^{(2)})$ hold, we have

$$\begin{aligned}&P\left(\overline{M}_n^{(1)}(R_n\setminus A_{\varepsilon_1})\le u_n^{(1)},\overline{M}_n^{(k)}(A_{\varepsilon_1})\le u_n^{(2)})P(\underline{M}_n^{(1)}(R_n\setminus A_{\varepsilon_2})>v_n^{(1)},\underline{M}_n^{(r)}(A_{\varepsilon_2})>v_n^{(2)}\right)\\=&P(\overline{M}_n^{(1)}(R_n\setminus A_{\varepsilon_1})\le u_n^{(1)})P(\overline{M}_n^{(k)}(A_{\varepsilon_1})\le u_n^{(2)})P(\underline{M}_n^{(1)}(R_n\setminus A_{\varepsilon_2})>v_n^{(1)})P(\underline{M}_n^{(r)}(A_{\varepsilon_2})>v_n^{(2)})\\&+o(1)\end{aligned}$$

which also converges to

$$e^{-\theta_2\tau_1'(1-\varepsilon_2)}\times e^{-\theta_1\tau_1(1-\varepsilon_1)}\times e^{-\theta_1\tau_2\varepsilon_1}\times\left[1+\sum_{i=1}^{k-1}\sum_{j=1}^{i}\frac{(\theta_1\tau_2\varepsilon_1)^j}{j!}\Pi_1^{*j}(i)\right]\times$$
$$\left[1-e^{-\theta_2\tau_2'\varepsilon_2}\left(1+\sum_{i=1}^{r-1}\sum_{j=1}^{i}\frac{(\theta_2\tau_2'\varepsilon_2)^j}{j!}\Pi_2^{*j}(i)\right)\right].$$

Proposition 3.1.: *Under the assumptions of Lemma 3.2. we have*

$$P\left(\bigcap_{i=1}^{k}\left\{\overline{L}_n^{(i)}\le n\varepsilon_1\right\},\bigcap_{i=1}^{r}\left\{\underline{L}_n^{(i)}\le n\varepsilon_2\right\}\right)-P\left(\bigcap_{i=1}^{k}\left\{\overline{L}_n^{(i)}\le n\varepsilon_1\right\}\right)P\left(\bigcap_{i=1}^{r}\left\{\underline{L}_n^{(i)}\le n\varepsilon_2\right\}\right)\xrightarrow[n\to\infty]{}0.$$

Proof: It follows from the relationship

$$\begin{aligned}&P\left(\bigcap_{i=1}^{k}\left\{\overline{L}_n^{(i)}\le n\varepsilon_1\right\},\bigcap_{i=1}^{r}\left\{\underline{L}_n^{(i)}\le n\varepsilon_2\right\}\right)\\ =\ &P\left(\overline{M}_n^{(1)}(R_n\setminus A_{\varepsilon_1})\le\overline{M}_n^{(k)}(A_{\varepsilon_1}),\underline{M}_n^{(1)}(R_n\setminus A_{\varepsilon_2})\ge\underline{M}_n^{(r)}(A_{\varepsilon_2})\right),\end{aligned}$$

and Lemma 3.2.

We finish this section by exhibiting a stationary sequence that verifies the assumptions of Proposition 3.1.

Example: Let $\{Y_n\}_{n\ge1}$ be a sequence of independent and identically distributed random variables with common distribution function, F, and define

$$X_n=max\left(Y_n,Y_{n+1}\right),\ n\ge1.$$

Let $\left\{u_n^{(i)}\right\}_{n\ge1}$ and $\left\{v_n^{(i)}\right\}_{n\ge1}$, $i=1,2$, be sequences of real numbers such that

$$n\left(1-F^2(u_n^{(i)})\right)\xrightarrow[n\to\infty]{}\tau_i\quad\text{and}\quad nF^2(-v_n^{(i)})\xrightarrow[n\to\infty]{}\tau_i'.$$

Then $\mathbf{X}$ and $-\mathbf{X}$ have extremal indexes $\theta_1=\frac{1}{2}$ and $\theta_2=1$, respectively. Moreover $\Pi_1(2)=\lim_{n\to\infty}\Pi_{n,1}(2)=1$ and $\Pi_2(1)=\lim_{n\to\infty}\Pi_{n,2}(1)=1$.

Since $\mathbf{X}$ is 2-dependent, for each $u_n^*\in\left\{u_n^{(1)},u_n^{(2)}\right\}$ and $v_n^*\in\left\{v_n^{(1)},v_n^{(2)}\right\}$, we obtain

$$\begin{aligned}&n\sum_{j=2}^{r_n}P\left(X_1>u_n^*,X_j\le-v_n^*\right)+P\left(X_1\le-v_n^*,X_j>u_n^*\right)\\ =\ &n\left(P\left(X_1>u_n^*,X_2\le-v_n^*\right)+P\left(X_1\le-v_n^*,X_2>u_n^*\right)\right)+\\ &n\sum_{j=3}^{r_n}P\left(X_1>u_n^*,X_j\le-v_n^*\right)+P\left(X_1\le-v_n^*,X_j>u_n^*\right)\\ \le\ &n\left(P(Y_1>u_n^*,Y_2\le-v_n^*,Y_3\le-v_n^*\right)+P\left(Y_1\le-v_n^*,Y_2\le-v_n^*,Y_3\ge u_n^*\right)\\ &+2nr_n\left(1-F^2\left(u_n^*\right)\right)F^2\left(-v_n^*\right)\\ \le\ &2n\left(1-F\left(u_n^*\right)\right)F^2\left(-v_n^*\right)+2nr_n\left(1-F^2\left(u_n^*\right)\right)F^2\left(-v_n^*\right)\xrightarrow[n\to\infty]{}0.\end{aligned}$$

References

Davis, R. (1982). Limit laws for the maximum and minimum of stationary sequences. *Z. Wharscheinlichkeits theorie verw. Gebiete*, 61, 31-42.

Davis, R. (1983). Limit laws for upper and lower extremes from stationary mixing sequences. *J. Multivariate Anal.*, 13, 273-286.

Davis, R. (1984). On upper and lower extremes in stationary sequences. *J. Tiago Oliveira (ed), Statistical Extremes and Applications*, 443-460.

Ferreira, H. and Scotto, M. (2002). On the asymptotic locations of high values of a stationary sequence. *Statistics and Probability Letters*, 60, 475-482.

Hsing, T., Husler, J. and Leadbetter, M.R. (1988). On the exceedance point process for a stationary sequence. *Prob. Rel. Fields*, 78, 97-112.

Nandagopalan, S. (1990). Multivariate extremes and estimation of the extremal index. *Ph. D. Thesis University of North Carolina at Chapel Hill*.

Leadbetter, M.R., Lindgren, G. and Rootzén, H. (1988) *Extremes and related properties of random sequences and processes*. Springer-Verlag, Berlim.

Pereira, L. and Ferreira, H. (2002) The asymptotic locations of the maximum and minimum of stationary sequences. *Journal of Statistical Planning and Inference*, 104, 287-295.