The asymptotic location of the maximum of a stationary random field

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Abstract: In this paper we study the limiting distribution of the location of the maximum generated by a stationary random field satisfying a long range weak dependence for each coordinate at a time.

Keywords: Location of maxima, exceedances, long range dependence, random field, extremal index.

1. INTRODUCTION

Let $\mathbf{X} = \left\{X_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^2\right\}$ be a random field on $\mathbb{N}^2$, where $\mathbb{N}$ is the set of all positive integers. For a family of real levels $\{u_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ and a subset $\mathbf{I}$ of the rectangle of points $\mathbf{R_n} = \{1,\ldots,n_1\}\times\{1,\ldots,n_2\}$, we will denote the event $\{X_{\mathbf{i}} \leq u_{\mathbf{n}} : \mathbf{i} \in \mathbf{I}\}$ by $M_{\mathbf{n}}(\mathbf{I})$ or simply by $M_{\mathbf{n}}$ when $\mathbf{I} = \mathbf{R_n}$. If $\mathbf{I} = \emptyset$ then $M_{\mathbf{n}}(\mathbf{I}) = -\infty$.

For each $i = 1,2$, we say the pair $\mathbf{I} \subset \mathbb{N}^2$ and $\mathbf{J} \subset \mathbb{N}^2$ is in $S_i(l)$ if the distance between $\Pi_i(\mathbf{I})$ and $\Pi_i(\mathbf{J})$ is greater or equal to l, where $\Pi_i, i = 1,2$, denote the cartesian projections.

Given a set of locations $\left\{\mathbf{j}^{(1)},\ldots,\mathbf{j}^{(n)}\right\}$, $\mathbf{j}^{(i)} = \left(j_1^{(i)}, j_2^{(i)}\right)$, for each location $\mathbf{j}^{(i)}$ let us considere the set of "predecessors" of $\mathbf{j}^{(i)}$, say $\mathcal{P}_{\mathbf{j}^{(i)}}$, as the set $\mathcal{P}_{\mathbf{j}^{(i)}} = \left\{\mathbf{j}^{(s)} : j_1^{(s)} \leq j_1^{(i)} \wedge j_2^{(s)} \leq j_2^{(i)}\right\} - \left\{\mathbf{j}^{(i)}\right\}$.

Let $\Im = \left\{\mathbf{j} \in \mathbb{N}^2 : X_{\mathbf{j}} = M_{\mathbf{n}}\right\}$, $\widetilde{\mathcal{P}} = \left\{\mathbf{j} \in \Im : \forall \mathbf{j}' \in \Im, \#\mathcal{P}_{\mathbf{j}} \leq \#\mathcal{P}_{\mathbf{j}'}\right\}$ and $L_{\mathbf{n}}$ the location of $M_{\mathbf{n}}$. We define

$$L_{\mathbf{n}} = \begin{cases} \mathbf{j}^{(1)} & \text{if } \Im = \left\{\mathbf{j}^{(1)}\right\} \\ \mathbf{j}^{(2)} & \text{if } \#\Im > 1 \wedge \widetilde{\mathcal{P}} = \left\{\mathbf{j}^{(2)}\right\} \\ \mathbf{j}^{(3)} & \text{if } \#\widetilde{\mathcal{P}} > 1 \wedge \mathbf{j}^{(3)} \in \Im \wedge \forall \mathbf{j}^{(s)} \in \Im, \mathbf{j}^{(s)} \neq \mathbf{j}^{(3)}, \Pi_1(\mathbf{j}^{(3)}) < \Pi_1(\mathbf{j}^{(s)}) \end{cases}.$$

We shall assume that $\mathbf{X}$ is a stationary random field and that there are constants $\{a_{\mathbf{n}} > 0\}_{\mathbf{n}\geq\mathbf{1}}$ and $\{b_{\mathbf{n}}\}_{\mathbf{n}\geq\mathbf{1}}$ such that, for each $x \in \mathbb{R}$,

$$P\left(a_{\mathbf{n}}^{-1}(M_{\mathbf{n}} - b_{\mathbf{n}}) \leq x\right) \underset{\mathbf{n}\to\infty}{\longrightarrow} H(x),$$

where H is a nondegenerate distribution function.

If $\mathbf{X}$ is a random field of independent and identically distributed random variables or if it satisfies the coordinatewise-mixing condition $\Delta(u_{\mathbf{n}}(x))$ from Leadbetter *et al.* (1988) (see also Choi, H. (2002)), with $u_{\mathbf{n}}(x) = a_{\mathbf{n}}x + b_{\mathbf{n}}$, then $\mathbf{X}$ verifies the Extremal Types Theorem, *id est*, G is Gumbel, Weibull or a Fréchet distribution.

Accordingly Choi, H. (2002), we shall say that $\mathbf{X}$ has extremal index θ, $0 \leq \theta \leq 1$, if for each $\tau > 0$ there exists $\left\{u_{\mathbf{n}}^{(\tau)}\right\}_{\mathbf{n}\geq\mathbf{1}}$ such that, as $\mathbf{n} \longrightarrow \infty$, $n_1 n_2 P\left(X_{\mathbf{1}} > u_{\mathbf{n}}^{(\tau)}\right) \longrightarrow \tau$ and $P\left(M_{\mathbf{n}} \leq u_{\mathbf{n}}^{(\tau)}\right) \longrightarrow$ $exp(-\theta\tau)$.

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Under local restrictions on the oscillations of the values of a random field, Ferreira, H. *et al.* (2005) and Pereira, L. *et al.* (2006) (see also Pereira, L. *et al.* (2005)) compute the extremal index of the random field from the joint distribution of a finite number of variables.

In this paper we show that the normalized location of the maximum of a stationary random field with extremal index $\theta \in]0,1]$ satisfying a slight generalization of $\Delta(u_{\mathbf{n}})-$condition converges to a uniform variable on $[0,1]^2$ and is asymptotically independent of the height of the maximum. We used the ideas, presented in Pereira, L. *et al.* (2002), to obtain the limiting distribution of the location of the maximum generated by a stationary sequence, with a specific approach for the random fields.

The result obtained allow us to select a set of observations of $\{X_{\mathbf{i}} : i \in \mathbf{R}_{\mathbf{n}}\}$, for example $\{X_{\mathbf{i}} : i \in \{1,\ldots,[n_1\varepsilon_1]\} \times \{1,\ldots,[n_2\varepsilon_2]\}\}$ with $\varepsilon_1, \varepsilon_2 \in (0,1]$, by ensuring that this set contains the maximum value of the stationary random field with a pre-determined probability.

2. LIMIT DISTRIBUTION OF THE LOCATION OF THE MAXIMUM GENERATED BY A STATIONARY RANDOM FIELD

We suppose that $\mathbf{X}$ satisfies a generalization of the coordinatewise-mixing condition $\Delta(u_{\mathbf{n}})$ introduced in Leadbetter *et al.* (1988), which exploits the past and future separation one coordinate at a time, and enable us to deal with the joint behavior of maxima over disjoint rectangles.

Definition 2.1. *Let $\mathbf{X}$ be a stationary random field and $\left\{u_{\mathbf{n}}^{(i)}\right\}_{\mathbf{n}\geq\mathbf{1}}, i=1,2,$ sequences of real numbers. The coordinatewise-mixing condition $\Delta_2(u_{\mathbf{n}}^{(1)}, u_{\mathbf{n}}^{(2)})$ is said to hold for $\mathbf{X}$ if there exist sequences of integer valued constants $\{k_{n_i}\}_{n_i\geq 1}$, $\{l_{n_i}\}_{n_i\geq 1}$, $i=1,2,$ such that, as $\mathbf{n}=(n_1,n_2) \longrightarrow \infty$, we have*

$$(2.1)\qquad (k_{n_1}, k_{n_2}) \longrightarrow \infty,\ \left(\frac{k_{n_1} l_{n_1}}{n_1}, \frac{k_{n_2} l_{n_2}}{n_2}\right) \longrightarrow \mathbf{0},\ \left(k_{n_1}\Delta^{(1)}_{\mathbf{n},l_{n_1}}, k_{n_1}k_{n_2}\Delta^{(2)}_{\mathbf{n},l_{n_2}}\right) \longrightarrow \mathbf{0},$$

where $\Delta^{(i)}_{\mathbf{n},l_{n_i}}, i=1,2,$ are the components of the mixing coefficient defined as follows:

$$\Delta^{(1)}_{\mathbf{n},l_{n_1}} = sup\left|P\left(M_{\mathbf{n}}(\mathbf{I}_1) \leq u_{\mathbf{n}}^{(i)^*}, M_{\mathbf{n}}(\mathbf{I}_2) \leq u_{\mathbf{n}}^{(i)^*}\right) - P\left(M_{\mathbf{n}}(\mathbf{I}_1) \leq u_{\mathbf{n}}^{(i)^*}\right) P\left(M_{\mathbf{n}}(\mathbf{I}_2) \leq u_{\mathbf{n}}^{(i)^*}\right)\right|,$$

where $u_{\mathbf{n}}^{(i)^*} \in \left\{u_{\mathbf{n}}^{(1)}, u_{\mathbf{n}}^{(2)}\right\}$ and the supremum is taken over pairs $\mathbf{I}_1$ and $\mathbf{I}_2$ in $S_1(l_{n_1})$ such that $|\Pi_1(\mathbf{I}_2)| \leq \frac{n_1}{k_{n_1}}$,

$$\Delta^{(2)}_{\mathbf{n},l_{n_2}} = sup\left|P\left(M_{\mathbf{n}}(\mathbf{I}_1) \leq u_{\mathbf{n}}^{(i)^*}, M_{\mathbf{n}}(\mathbf{I}_2) \leq u_{\mathbf{n}}^{(i)^*}\right) - P\left(M_{\mathbf{n}}(\mathbf{I}_1) \leq u_{\mathbf{n}}^{(i)^*}\right) P\left(M_{\mathbf{n}}(\mathbf{I}_2) \leq u_{\mathbf{n}}^{(i)^*}\right)\right|,$$

where $u_{\mathbf{n}}^{(i)^*} \in \left\{u_{\mathbf{n}}^{(1)}, u_{\mathbf{n}}^{(2)}\right\}$ and the supremum is taken over pairs $\mathbf{I}_1$ and $\mathbf{I}_2$ in $S_2(l_{n_2})$ such that $\Pi_1(\mathbf{I}_1) = \Pi_1(\mathbf{I}_2)$ and $|\Pi_2(\mathbf{I}_2)| \leq \frac{n_2}{k_{n_2}}$.

For $u_{\mathbf{n}}^{(1)} \equiv u_{\mathbf{n}}^{(2)} \equiv u_{\mathbf{n}}$ condition $\Delta_2(u_{\mathbf{n}}^{(1)}, u_{\mathbf{n}}^{(2)})$ reduces to the coordinatewise-mixing condition $\Delta(u_{\mathbf{n}})$ (Leadbetter *et al.* (1988) and Choi, H. (2002)).

Lemma 2.1.: *Let $\left\{u_{\mathbf{n}}^{(i)}\right\}_{\mathbf{n}\geq\mathbf{1}}, i=1,2,$ be sequences of real numbers such that*

$$(2.2)\qquad n_1 n_2 P\left(X_{\mathbf{1}} > u_{\mathbf{n}}^{(i)}\right) \underset{\mathbf{n}\to\infty}{\longrightarrow} \tau_i,\ i=1,2,$$

where $\tau_1, \tau_2 < \infty$. If the stationary random field $\mathbf{X}$ satisfies $\Delta_2(u_{\mathbf{n}}^{(1)}, u_{\mathbf{n}}^{(2)})$ for sequences $\{k_{n_i}\}_{n_i\geq 1}$, $\{l_{n_i}\}_{n_i\geq 1}$, $\left\{u_{\mathbf{n}}^{(i)}\right\}_{\mathbf{n}\geq\mathbf{1}}, i=1,2,$ satisfying (2.1), and the rectangles $\mathbf{V}_{s,t} \subset \mathbf{R}_{\mathbf{n}}$, $s=1,\ldots,k_{n_1}$ and

$t=1,\ldots,k_{n_2}$, are disjoint, then

$$\left|P\left(\bigcap_{s=1}^{k_{n_1}}\bigcap_{t=1}^{k_{n_2}}\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\right)-\prod_{s=1}^{k_{n_1}}\prod_{t=1}^{k_{n_2}}P\left(\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\right)\right|\xrightarrow[\mathbf{n}\to\infty]{}0,$$

where, for each s and t, $u_{\mathbf{n},s,t}$ is any one of $u_{\mathbf{n}}^{(1)},u_{\mathbf{n}}^{(2)}$.

Proof: From (2.1) and (2.2), for the purpose of the above convergence, we can assume that $\Pi_1(\mathbf{V}_{s,t})>l_{n_1}$ or $\Pi_2(\mathbf{V}_{s,t})>l_{n_2}$. If all the pairs of rectangles $\mathbf{V}_{s,t}$ are in $S_1(l_{n_1})\cup S_2(l_{n_2})$ then we have

$$\begin{aligned}
&\left|P\left(\bigcap_{s=1}^{k_{n_1}}\bigcap_{t=1}^{k_{n_2}}\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)-\prod_{s=1}^{k_{n_1}}\prod_{t=1}^{k_{n_2}}P\left(\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)\right|\\
\le&\left|P\left(\bigcap_{s=1}^{k_{n_1}}\bigcap_{t=1}^{k_{n_2}}\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)-\prod_{s=1}^{k_{n_1}}P\left(\bigcap_{t=1}^{k_{n_2}}\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)\right|\\
&+\left|\prod_{s=1}^{k_{n_1}}P\left(\bigcap_{t=1}^{k_{n_2}}\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)-\prod_{s=1}^{k_{n_1}}\prod_{t=1}^{k_{n_2}}P\left(\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)\right|\\
\le&\sum_{j=1}^{k_{n_1}-1}\left|P\left(\bigcap_{s=j}^{k_{n_1}}\bigcap_{t=1}^{k_{n_2}}\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)\prod_{s=1}^{j-1}P\left(\bigcap_{t=1}^{k_{n_2}}\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)-\right.\\
&\left.P\left(\bigcap_{s=j+1}^{k_{n_1}}\bigcap_{t=1}^{k_{n_2}}\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)\prod_{s=1}^{j}P\left(\bigcap_{t=1}^{k_{n_2}}\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)\right|\\
&+\sum_{s=1}^{k_{n_1}}\sum_{j=1}^{k_{n_2}-1}\left|P\left(\bigcap_{t=j}^{k_{n_2}}\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)\prod_{t=1}^{j-1}P\left(\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)-\right.\\
&\left.P\left(\bigcap_{t=j+1}^{k_{n_2}}\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)\prod_{t=1}^{j}P\left(\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)\right|\\
\le&\sum_{j=1}^{k_{n_1}-1}\left|P\left(\bigcap_{s=j}^{k_{n_1}}\bigcap_{t=1}^{k_{n_2}}\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)-P\left(\bigcap_{s=j+1}^{k_{n_1}}\bigcap_{t=1}^{k_{n_2}}\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)\times\right.\\
&\left.P\left(\bigcap_{\mathbf{i}\in\mathbf{V}_{j,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)\right|\\
&+\sum_{s=1}^{k_{n_1}}\sum_{j=1}^{k_{n_2}-1}\left|P\left(\bigcap_{t=j}^{k_{n_2}}\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)-P\left(\bigcap_{t=j+1}^{k_{n_2}}\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)\times\right.\\
&\left.P\left(\bigcap_{\mathbf{i}\in\mathbf{V}_{s,j}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)\right|\\
\le&\ k_{n_1}\Delta^{(1)}_{\mathbf{n},l_{n_1}}+k_{n_1}k_{n_2}\Delta^{(2)}_{\mathbf{n},l_{n_2}}=o(1).
\end{aligned}$$

If some pair of rectangles $\mathbf{V}_{s,t}$ are not in $S_1(l_{n_1})\cup S_2(l_{n_2})$ we can eliminate l_{n_1} columns or l_{n_2} rows in $\mathbf{V}_{s,t}$ in order to obtain $\mathbf{V}^*_{s,t}\subset\mathbf{V}_{s,t}$, $s=1,\ldots,k_{n_1}$, $t=1,\ldots,k_{n_2}$, and we obtain

$$\begin{aligned}
&\left|P\left(\bigcap_{s=1}^{k_{n_1}}\bigcap_{t=1}^{k_{n_2}}\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)-\prod_{s=1}^{k_{n_1}}\prod_{t=1}^{k_{n_2}}P\left(\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)\right|\\
\le&\left|P\left(\bigcap_{s=1}^{k_{n_1}}\bigcap_{t=1}^{k_{n_2}}\bigcap_{\mathbf{i}\in\mathbf{V}_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)-P\left(\bigcap_{s=1}^{k_{n_1}}\bigcap_{t=1}^{k_{n_2}}\bigcap_{\mathbf{i}\in\mathbf{V}^*_{s,t}}\{X_{\mathbf{i}}\le u_{\mathbf{n},s,t}\}\right)\right|+
\end{aligned}$$

$$
\begin{aligned}
& \left| P\left(\bigcap_{s=1}^{k_{n_1}} \bigcap_{t=1}^{k_{n_2}} \bigcap_{\mathbf{i}\in \mathbf{V}^*_{s,t}} \{X_{\mathbf{i}} \le u_{\mathbf{n},s,t}\} \right) - \prod_{s=1}^{k_{n_1}} \prod_{t=1}^{k_{n_2}} P\left(\bigcap_{\mathbf{i}\in \mathbf{V}^*_{s,t}} \{X_{\mathbf{i}} \le u_{\mathbf{n},s,t}\} \right) \right| + \\
& \left| \prod_{s=1}^{k_{n_1}} \prod_{t=1}^{k_{n_2}} P\left(\bigcap_{\mathbf{i}\in \mathbf{V}^*_{s,t}} \{X_{\mathbf{i}} \le u_{\mathbf{n},s,t}\} \right) - \prod_{s=1}^{k_{n_1}} \prod_{t=1}^{k_{n_2}} P\left(\bigcap_{\mathbf{i}\in \mathbf{V}_{s,t}} \{X_{\mathbf{i}} \le u_{\mathbf{n},s,t}\} \right) \right| \\
\le \;& 2k_{n_1}k_{n_2}l_{n_1}l_{n_2} max\left(P\left(X_{\mathbf{1}} > u_{\mathbf{n}}^{(1)}\right), P\left(X_{\mathbf{1}} > u_{\mathbf{n}}^{(2)}\right)\right) + k_{n_1}k_{n_2}\Delta^{(2)}_{\mathbf{n},l_{n_2}} = o(1).
\end{aligned}
$$

The next Lemma shows that, for each $\varepsilon_1, \varepsilon_2 \in (0,1]$, the events $\left\{M_{\mathbf{n}}\left([1,n_1\varepsilon_1]\times[1,n_2\varepsilon_2]\cap\mathbb{N}^2\right) \le u_{\mathbf{n}}^{(1)}\right\}$ and $\left\{M_{\mathbf{n}}\left(\mathbf{R}_{\mathbf{n}} \setminus \left([1,n_1\varepsilon_1]\times[1,n_2\varepsilon_2]\cap\mathbb{N}^2\right)\right) \le u_{\mathbf{n}}^{(2)}\right\}$ are asymptotically independent, and is the key to obtain the limiting distribution of the location of maximum. It follows as a consequence of Lemma 2.1..

Lemma 2.2.: *Suppose that the stationary random field* $\mathbf{X}$ *satisfies* $\Delta_2(u_{\mathbf{n}}^{(1)}, u_{\mathbf{n}}^{(2)})$, *where the levels* $u_{\mathbf{n}}^{(i)}, i = 1, 2$, *satisfy (2.2). Then, for each* $\varepsilon_1, \varepsilon_2 \in (0,1]$,

$$
\begin{aligned}
& P\left(M_{\mathbf{n}}\left([1,n_1\varepsilon_1]\times[1,n_2\varepsilon_2]\cap\mathbb{N}^2\right) \le u_{\mathbf{n}}^{(1)}, M_{\mathbf{n}}\left(\mathbf{R}_{\mathbf{n}} \setminus \left([1,n_1\varepsilon_1]\times[1,n_2\varepsilon_2]\cap\mathbb{N}^2\right)\right) \le u_{\mathbf{n}}^{(2)}\right) \\
& -P\left(M_{\mathbf{n}}\left([1,n_1\varepsilon_1]\times[1,n_2\varepsilon_2]\cap\mathbb{N}^2\right) \le u_{\mathbf{n}}^{(1)}\right) P\left(M_{\mathbf{n}}\left(\mathbf{R}_{\mathbf{n}} \setminus \left([1,n_1\varepsilon_1]\times[1,n_2\varepsilon_2]\cap\mathbb{N}^2\right)\right) \le u_{\mathbf{n}}^{(2)}\right) \to 0,
\end{aligned}
$$

as $\mathbf{n} \to \infty$.

We finish by proving that the normalized location of the maximum is asymptotically uniform and independent of its height.

Proposition 2.1.: *Let* $\mathbf{X}$ *be a stationary random field with extremal index* $0 < \theta \le 1$ *and* $\{a_{\mathbf{n}} > 0\}_{\mathbf{n}\ge\mathbf{1}}$ *and* $\{b_{\mathbf{n}}\}_{\mathbf{n}\ge\mathbf{1}}$ *sequences of real numbers such that*

$$
P\left(M_{\mathbf{n}} \le a_{\mathbf{n}}x + b_{\mathbf{n}}\right) \xrightarrow[\mathbf{n}\to\infty]{} G^{\theta}(x),
$$

with a nondegenerate distribution function G.
If for $x_1, x_2 \in \mathbb{R}$ *and* $u_{\mathbf{n}}^{(i)} = u_{\mathbf{n}}(x_i) = a_{\mathbf{n}}x_i + b_{\mathbf{n}}$, $i = 1, 2$, $\mathbf{X}$ *satisfies the condition* $\Delta_2(u_{\mathbf{n}}^{(1)}, u_{\mathbf{n}}^{(2)})$ *then, for each* $\varepsilon_1, \varepsilon_2 \in (0,1]$,

$$
P\left(L_{\mathbf{n}} \in \left([1,n_1\varepsilon_1]\times[1,n_2\varepsilon_2]\cap\mathbb{N}^2\right), a_{\mathbf{n}}^{-1}(M_{\mathbf{n}} - b_{\mathbf{n}}) \le x\right) \xrightarrow[\mathbf{n}\to\infty]{} \varepsilon_1\varepsilon_2 G^{\theta}(x).
$$

Proof: For each $\varepsilon_1, \varepsilon_2 \in (0,1]$, it holds

$$
\begin{aligned}
& P\left(L_{\mathbf{n}} \in \left([1,n_1\varepsilon_1]\times[1,n_2\varepsilon_2]\cap\mathbb{N}^2\right), a_{\mathbf{n}}^{-1}(M_{\mathbf{n}} - b_{\mathbf{n}}) \le x\right) \\
= \;& P\left(M_{\mathbf{n}}\left([1,n_1\varepsilon_1]\times[1,n_2\varepsilon_2]\cap\mathbb{N}^2\right) \ge M_{\mathbf{n}}\left(\mathbf{R}_{\mathbf{n}} \setminus \left([1,n_1\varepsilon_1]\times[1,n_2\varepsilon_2]\cap\mathbb{N}^2\right)\right), M_{\mathbf{n}} \le a_{\mathbf{n}}x + b_{\mathbf{n}}\right) \\
= \;& P\left(M_{\mathbf{n}}\left(\mathbf{R}_{\mathbf{n}} \setminus \left([1,n_1\varepsilon_1]\times[1,n_2\varepsilon_2]\cap\mathbb{N}^2\right)\right) \le M_{\mathbf{n}}\left([1,n_1\varepsilon_1]\times[1,n_2\varepsilon_2]\cap\mathbb{N}^2\right) \le a_{\mathbf{n}}x + b_{\mathbf{n}}\right)
\end{aligned}
$$

By applying Lemma 2.2. with $x_i \in \mathbb{R}$ and $u_{\mathbf{n}}^{(i)} = a_{\mathbf{n}}x_i + b_{\mathbf{n}}$, $i = 1, 2$, the above probability converges to $P\left(V^{(1)} \le U^{(1)} \le x\right)$, where $U^{(1)}$ and $V^{(1)}$ are independent random variables whose distributions can be obtained as follows:
By attending that

$$
\begin{aligned}
& P\left(M_{\mathbf{n}}\left(\mathbf{R}_{\mathbf{n}} \setminus \left([1,n_1\varepsilon_1]\times[1,n_2\varepsilon_2]\cap\mathbb{N}^2\right)\right) \le u_{\mathbf{n}}(t)\right) \\
= \;& P\left(M_{\mathbf{n}}\left(\{1,\ldots,[n_1(1-\varepsilon_1)]\}\times\{1,\ldots,n_2\}\right) \le u_{\mathbf{n}}(t)\right) \times \\
& P\left(M_{\mathbf{n}}\left(\{1,\ldots,[n_1\varepsilon_1)]\}\times\{1,\ldots,[n_2(1-\varepsilon_2)]\}\right) \le u_{\mathbf{n}}(t)\right) + o(1),
\end{aligned}
$$

$\mathbf{X}$ has extremal index θ and, for each $t \in \mathbb{R}$,

$$[n_1(1-\varepsilon_1)]\, n_2 P\left(X_{\mathbf{1}} > u_{\mathbf{n}}(t)\right) \xrightarrow[\mathbf{n}\to\infty]{} -(1-\varepsilon_1)\log G(t),$$
$$[n_1\varepsilon_1]\,[n_2(1-\varepsilon_2)]\, P\left(X_{\mathbf{1}} > u_{\mathbf{n}}(t)\right) \xrightarrow[\mathbf{n}\to\infty]{} -\varepsilon_1(1-\varepsilon_2)\log G(t),$$

and

$$[n_1\varepsilon_1]\,[n_2\varepsilon_2]\, P\left(X_{\mathbf{1}} > u_{\mathbf{n}}(t)\right) \xrightarrow[\mathbf{n}\to\infty]{} -\varepsilon_1\varepsilon_2\log G(t),$$

then

$$P\left(V^{(1)} \leq t\right) = \mathop{lim}_{\mathbf{n}\to\infty} P\left(M_{\mathbf{n}}\left(\mathbf{R}_{\mathbf{n}} \setminus \left([1, n_1\varepsilon_1] \times [1, n_2\varepsilon_2] \cap \mathbb{N}^2\right)\right) \leq u_{\mathbf{n}}(t)\right) = G^{(1-\varepsilon_1\varepsilon_2)\theta}(t), \tag{2.3}$$

and

$$P\left(U^{(1)} \leq t\right) = \mathop{lim}_{\mathbf{n}\to\infty} P\left(M_{\mathbf{n}}\left([1, n_1\varepsilon_1] \times [1, n_2\varepsilon_2] \cap \mathbb{N}^2\right) \leq u_{\mathbf{n}}(t)\right) = G^{\varepsilon_1\varepsilon_2\theta}(t). \tag{2.4}$$

Therefore, from (2.3) and (2.4), we get

$$\begin{aligned}
& \lim_{\mathbf{n}\to\infty} P\left(L_{\mathbf{n}} \in \left([1, n_1\varepsilon_1] \times [1, n_2\varepsilon_2] \cap \mathbb{N}^2\right), a_{\mathbf{n}}^{-1}(M_{\mathbf{n}} - b_{\mathbf{n}}) \leq x\right) \\
= \; & P\left(V^{(1)} \leq U^{(1)} \leq x\right) \\
= \; & \int_{]-\infty, x]} G^{\theta(1-\varepsilon_1\varepsilon_2)}(t)\, dG^{\varepsilon_1\varepsilon_2}(t) \\
= \; & \varepsilon_1\varepsilon_2 G^{\theta}(x).
\end{aligned}$$

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