On the extremal behaviour of a simple and a power max-autoregressive model

Marta Ferreira

Universidade do Minho, Dep. Matemática, CMAT

Luísa Canto e Castro

Faculdade de Ciências da Universidade de Lisboa, DEIO-CEAUL

Abstract: Nowadays everyone is aware of the importance of studying the maximums and exceedances in various areas like hydrology, geophysics or finances. These phenomenon are often associated with markovian sequences for which we claim a simple treatment in what concerns the extremes topic. A very simple example that can be pointed out is a *max-autoregressive* process ($ARMAX$). It can be found in Alpuim [**1**] and Canto e Castro [**2**] some dependence and extremal performances of different versions of an $ARMAX$ process. In this work, we first consider an $ARMAX$ in its simplest form and then, motivated by the Ledford and Tawn coefficient of tail dependence (η), we present a new max-autoregressive process involving a power transformation which we call $ARMAX$-*p*. We analyze some local dependence conditions and check some specific requisites in order to establish the limit behavior of the tail empirical quantile function and hence the asymptotic normality of the Hill estimator according to Drees [**9**].

keywords Statistic of extremes, Stationary sequence, Dependence conditions, Tail index, Extremal index, Tail dependence index, Hill estimator.

1. Introduction

Extreme Value Theory (EVT) became widely used in many applied sciences as they were faced with modeling high values. Ocean wave modeling, wind engineering, thermodynamics of earthquakes, risk assessment on financial markets are some examples. These phenomenon are often associated with markovian time series and in this paper we have selected two versions of *max-autoregressive process* ($ARMAX$), since they have a nice behavior in what concerns extremes inference.

In section 1.1 we enounce all the results used along this paper. We start with some well-known results of classical EVT and we include two very specific subsections. In the first one (subsection 1.2) we display some assertions of Ledford and Tawn model ([**12**], [**13**]), considering only the bivariate case, in which appears a new parameter, η, named as tail dependence coefficient. Subsection 1.3 exhibits a result from Drees (Drees [**9**], Theorem 2.1) in which it is stated the limit behavior of the tail empirical quantile function and hence the asymptotic normality of a class of estimators that includes the Hill estimator.

The section 2 is all devoted to a very simple formulation of an $ARMAX$ process. We analyze its extremal behavior under some imposed conditions and check its local dependence structure. We also prove that it fulfils the conditions of the above mentioned Theorem 2.2 in Drees [**9**] and we calculate the value of Ledford and Tawn tail dependence index, η, considering the bivariate random vector, (X_1, X_{1+m}). These calculations motivated a new formulation of $ARMAX$ process which involves a power transformation and so named as *power max-autoregressive process* ($ARMAX$-p). We shall see in section 3 that η depends on the value of $ARMAX$-p parameter. In this section we also analyze the extremal behavior of this new process and its local dependence structure as well. An application of Theorem 2.2 from Drees [**9**] will be the next step of our future work.

1.1. Extreme Value Theory under independence and local dependence

In the present paper $\{\hat{X}_n, n \geq 1\}$ will be an i.i.d. sequence of random variables (r.v.'s) with common distribution function (d.f.) F. We start by enouncing the central result of classical Extreme Value Theory - The Extremal Types Theorem.

Theorem 1.1. *Let $\hat{M}_n = \max(\hat{X}_1, ..., \hat{X}_n)$, where $\hat{X}_1, \hat{X}_2, ...$ are i.i.d. random variables. If for some constants $a_n > 0$, b_n, we have*

$$P(\hat{M}_n \leq a_n x + b_n) \underset{n\to\infty}{\longrightarrow} G(x) \tag{1}$$

for some non-degenerate G, then G is one of the three extreme value types distributions

$$\text{Type I (Gumbel)}: \quad G(x) = exp(-e^{-x}) \ , \quad -\infty < x < \infty;$$

$$\text{Type II (Fréchet)}: \quad G(x) = \begin{cases} 0 & , x \leq 0 \\ exp(-x^{-\alpha}) \ , \text{ for some } \alpha > 0 & , x > 0; \end{cases}$$

$$\text{Type III (Weibull)}: \quad G(x) = \begin{cases} exp(-(-x)^{\alpha}) \ , \text{ for some } \alpha > 0 & , x \leq 0 \\ 1 & , x > 0. \end{cases}$$

Conversely, each d.f. G of extreme value type may appear as a limit in (1). G can be uniquely represented in the following parametric way, known as Generalized Extreme Value function (GEV),

$$G_\gamma(x) = exp(-(1+\gamma x)^{-1/\gamma}), \ 1+\gamma x > 0, \ \gamma \in \mathbb{R}, \tag{2}$$

which is interpreted as $exp(-e^{-x})$ if $\gamma = 0$. The parameter γ is the tail index with $\gamma > 0$ corresponding to the Fréchet case and $\gamma < 0$ corresponding to the Weibull case .

We say that F is in the domain of attraction of G_γ, denoted by $F \in \mathcal{D}(G_\gamma)$, if the limit (1) is verified for some real constants $a_n > 0$ e b_n. The Extremal Types Theorem shows the crucial importance of the tail index in the Extreme Value Theory (EVT) as it determines, in a uniquely way, the type of limiting distribution of the normalized maxima. Another important result gives a simple necessary and sufficient condition under which $P(\hat{M}_n \leq u_n)$ converges, for a given sequence of constants $\{u_n\}$. It also plays an important role in dependent cases as we shall see.

Theorem 1.2. *Let $0 \leq \tau \leq \infty$ and suppose that $\{u_n\}$ is a real sequence such that*

$$n(1 - F(u_n)) \to \tau \text{ as } n \to \infty. \tag{3}$$

Then,

$$P(\hat{M}_n \leq u_n) \to e^{-\tau} \text{ as } n \to \infty. \tag{4}$$

Conversely, if (4) holds for some τ, $0 \leq \tau \leq \infty$, then so does (3).

If (3) holds for some $\tau \geq 0$, we say that u_n is a normalized level and it is denoted as $u_n(\tau)$.
From now on, we will consider a stationary sequence of r.v.'s $\{X_n, n \geq 1\}$ with the same common d.f. F considered above for the i.i.d. r.v.'s $\hat{X}_1, \hat{X}_2, ...$, which exhibits a specified dependence structure but not too strong. In fact, we will assume that the dependence between X_i e X_j falls off in some specified way as $|i - j|$ increases.

Let us start by the simplest type of dependence considered here (Watson [**15**]):

Definition 1.1. *The sequence $\{X_i\}$ is m-dependent, if the r.v.'s X_i e X_j are independent whenever $|i - j| > m$.*

A more commonly used dependence restriction of this type for stationary sequences is that of *strong mixing.*

Definition 1.2. *A sequence $\{X_i\}$ is said to satisfy the strong mixing assumption, if there is a function $g(k)$, the "mixing function", tending to zero as $k \to \infty$ and such that*

$$|P(A \cap B) - P(A)P(B)| < g(k),$$

when $A \in \mathcal{F}(X_1, ..., X_p)$ and $B \in \mathcal{F}(X_{p+k+1}, X_{p+k+2}, ...)$ for any p and k, with $\mathcal{F}(.)$ denoting the $\sigma-$field generated by the indicated random variables.

A less restrictive but also very used is the *β-mixing condition.*

Definition 1.3. *We say that a stationary sequence $\{X_i\}$ is β-mixing if*

$$\beta(k) := \sup_{p \in \mathbb{N}} E\big(\sup_{B \in \mathcal{F}(X_{p+k+1}, ...)} |P(B|\mathcal{F}(X_1, ..., X_p)) - P(B)|\big) \underset{k \to \infty}{\longrightarrow} 0.$$

Next, we are going to define the dependence conditions, D, D′ e D″, presented in Leadbetter, *et al.* [**10**].

Let $M_n = \max(X_1, ..., X_n)$. For brevity, we will use the following notation:

$$P(X_{i_1} \leq u_n, ..., X_{i_m} \leq u_n) = F_{i_1, ..., i_m}(u_n, ..., u_n) = F_{i_1, ..., i_m}(u_n),$$

for any integers $i_1 < ... < i_m$ and $\{u_n\}$ a given real sequence.

Definition 1.4. *Condition $D(u_n)$ will be said to hold if for any integers $\{i_1, ... i_p\}$ and $\{j_1, ... j_{p'}\}$, such that,*

$$1 \leq i_1 < ... < i_p < j_1 < ... < j_{p'} \leq n \text{ with } j_1 - i_p \geq \ell,$$

we have

$$|F_{i_1, ... i_p j_1, ... j_{p'}}(u_n) - F_{i_1, ... i_p}(u_n) F_{j_1, ... j_{p'}}(u_n)| \leq \alpha_{n, \ell} \tag{5}$$

where $\alpha_{n, \ell_n} \to 0$ as $n \to \infty$ for some sequence $\ell_n = o(n)$.

Note that this condition is a weaker version of the mixing condition. In extreme value theory, the events of interest are typically those of the form $\{X_i \leq u\}$ or their intersections. So $\mathrm{D}(u_n)$ is like a mixing condition but only required to hold for events of this type.

The next result states that the Extremal Types Theorem (1.1) holds for stationary sequences under condition $\mathrm{D}(u_n)$:

Theorem 1.3. *Let $\{X_i\}$ be a stationary sequence and $\{a_n > 0\}$ and $\{b_n \in \mathbb{R}\}$ given constants such that $P\big(M_n \leq a_n x + b_n\big)$ converges to a non-degenerate d.f. $G(x)$. Suppose that $D(u_n)$ is satisfied for $u_n = a_n x + b_n$ for each $x \in \mathbb{R}$. Then $G(x)$ is a GEV distribution, defined in (2).*

The results given so far have been concerned with the possible forms of limiting extreme value distributions. We now turn to the existence of such a limit, in that we formulate conditions under which $n(1-F(u_n)) \to \tau$ as $n \to \infty$ are equivalent for stationary sequences. In a dependence scenario restricted to the condition $\mathrm{D}(u_n)$ it can be stated the following sufficient result:

Theorem 1.4. *Suppose $u_n(\tau)$ is defined for $\tau > 0$, such that $n(1 - F(u_n(\tau))) \to \tau$ and that $D(u_n(\tau))$ holds for each such τ. If $P\big(M_n \leq u_n(\tau_*)\big)$ converges for some $\tau_* > 0$ then converges for all $\tau > 0$ and*

$$P\big(M_n \leq u_n(\tau)\big) \underset{n\to\infty}{\longrightarrow} e^{-\theta\tau}.$$

A new parameter comes out, θ, and it is known as *extremal index.* In Leadbetter *et al.* [**10**] it is defined as follows:

Definition 1.5. *We shall say that the process $\{X_i\}$ has extremal index θ ($\theta \in [0,1]$) if, for each $\tau > 0$,*

(i) *there exists $u_n(\tau)$ such that $n(1 - F(u_n(\tau))) \to \tau$,*
(ii) *$P(M_n \leq u_n(\tau)) \to e^{-\theta\tau}$.*

We conclude immediately that i.i.d. sequences whose maximum converges to a non-degenerate limiting distribution have unit extremal index. It may be seen in Leadbetter *et al.* [**10**] that if (3) holds then condition $\mathrm{D}(u_n)$ is also sufficient to guarantee that $\liminf P(M_n \leq u_n) \geq e^{-\tau}$ but a further assumption is needed to obtain the opposite inequality for the upper limit. So, they introduced another dependence condition, $\mathrm{D}'(u_n)$, defined in the following way:

Definition 1.6. *Condition $D'(u_n)$ will be said to hold for the stationary sequence $\{X_n\}$ and sequence u_n of constants if*

$$\limsup_{n\to\infty} n \sum_{j=2}^{r_n} P\big(X_1 > u_n, X_j > u_n\big) = 0, \text{ as } k \to \infty\,,$$

where $r_n = [n/k]$ and $[\,]$ denotes the integer part.

The condition $\mathrm{D}'(u_n)$ bounds the probability of more than one exceedance of u_n among $X_1, ...X_{r_n}$ and so, such exceedances tend to behave as the ones in an i.i.d. context. Therefore, M_n and $\hat{M}_n$ have a similar asymptotic behavior and so, θ assumes an unit value.

The next result states the already mentioned generalization of Theorem (1.2) to stationary sequences under conditions $\mathrm{D}(u_n)$ and $\mathrm{D}'(u_n)$:

Theorem 1.5. *Let $\{u_n\}$ be constants such that $D(u_n)$ and $D'(u_n)$ hold for the stationary sequence $\{X_n\}$. Let $\tau \in [0,\infty)$. Then,*

$$P\big(M_n \leq u_n\big) \to e^{-\tau} \text{ if and only if } n(1-F(u_n)) \to \tau.$$

It may also happened that the exceedances of u_n in a stationary sequence appear in clusters and so condition $\mathrm{D}'(u_n)$ does not hold. Leadbetter and Nandagopalan [**11**] have also studied such sequences and presented another local dependence condition, $\mathrm{D}''(u_n)$, which restricts rapid oscillations near high levels.

Definition 1.7. *Condition $D''(u_n)$ is said to be hold if $D(u_n)$ holds with mixing coefficients α_{n,l_n}, k_n are integers such that $k_n \to \infty$, $k_n\alpha_{n,l_n} \to 0$, $k_n l_n/n \to 0$ and $k_n(1-F(u_n)) \to 0$ and*

$$\lim_{n\to\infty} n \sum_{j=2}^{[n/k_n]-1} P\big(X_1 > u_n, X_j \leq u_n < X_{j+1}\big) = 0. \tag{6}$$

This condition involves a weaker restriction than $\mathrm{D}'(u_n)$ which is used to guarantee that $\theta = 1$, whereas under D'' all values of $\theta \in (0,1]$ are possible.

The next result states that, under the weak dependence conditions D and D'', the joint distribution of X_1 and X_2 determines whether the extremal index exists and gives its value.

Theorem 1.6. *Suppose $u_n(\tau)$ is defined for $\tau > 0$ and is such that $n(1-F(u_n(\tau))) \to \tau$. If $D(u_n(\tau))$ and $D''(u_n(\tau))$ holds for all positive τ and for some $\tau_* > 0$*

$$\lim_{n\to\infty} P\big(X_2 \leq u_n(\tau_*)|X_1 > u_n(\tau_*)\big) = \theta \ ,$$

then it holds for all positive τ and the process has extremal index equal to θ.

1.2. Ledford and Tawn coefficient of tail dependence: the parameter η

Now we turn ourselves to the multivariate field, specifically the bivariate case.
In the classical multivariate EVT we cannot distinguish asymptotically between exact independence of the components of a random vector and a moderate dependence which vanishes as the observations go more extreme. If we take, for instance, a bivariate normal random vector with correlation $\rho < 1$, we verify that it has the same limit distribution for the standardized maxima of the components independently of the value of ρ.
In order to overcome this problem, Ledford and Tawn ([**12**], [**13**]) proposed a model, where the penultimate tail dependence is characterized by the so-called coefficient of tail dependence, $\eta \in (0,1]$, which measures the dependence between the marginal tails.

Suppose that (X_i, Y_i), $i = 1, ..., n$, is a sequence of i.i.d. random vectors with common d.f. F and marginal d.f.'s F_1 and F_2. Considering $U = 1 - F_1(X)$ and $V = 1 - F_2(Y)$, the basic Ledford and Tawn model assumption can be reformulated as (Draisma *et al.*, [**5**]),

$$P\Big(\frac{U}{t} < x, \frac{V}{t} < y)|U < t, V < t\Big) = \frac{P(U < tx, V < ty)}{P(U < t, V < t)} \to c(x,y) \ , \text{ as } t \downarrow 0, \tag{7}$$

uniformly on $\{(x,y)|\max(x,y)=1\}$ for some non-degenerate function c. It is assumed that the function $t\mapsto P(U<t,V<t)$ is regularly varying at 0 with index $1/\eta$ and that $l:=\lim_{t\downarrow 0}P(U<t|V<t)$ exists. Note that we can also write l as the $\lim_{t\downarrow 0}P(U<t,V<t)/t=\lim_{t\downarrow 0}t^{1/\eta-1}L(t)$, for some slowly varying function L. So, we have $l=0$ if $\eta<1$ and $l>0$ if the marginals are asymptotically dependent providing that $L(t)\not\to 0$ as $t\to\infty$. Roughly speaking, four distinct cases can be considered: $\eta=1$ in case of asymptotic dependence, $\eta\in(1/2,1)$ and $\eta\in(1/2,1)$ if it exists, respectively, positive and negative dependence, both vanishing asymptotically, and $\eta=1/2$ in case of independence.

The function c is homogeneous of order $1/\eta$, since $c(tx,ty)=t^{1/\eta}c(x,y)$.

1.3. Tail empirical process under dependence

Using a weighted approximation of the tail empirical quantile function (q.f.), $Q_n(t):=X_{n-[k_nt]:n}$, where $\{k_n\}$ is an intermediate sequence, that is, a sequence of integers such that $k_n\to\infty$ and $k_n/n\to 0$, as $n\to\infty$, for stationary *β-mixing* time series, Drees [**8**] stated its asymptotic behavior under the following conditions:

- a regularity condition for the joint tail of (X_1,X_{1+m}):

$$\lim_{n\to\infty}\frac{n}{k_n}P\Big(X_1>F^{-1}\big(1-\frac{k_n}{n}x\big),X_{1+m}>F^{-1}\big(1-\frac{k_n}{n}y\big)\Big)\to c_m(x,y),\tag{8}$$

  for all $m\in\mathbb{N}$, $0<x,y\le 1+\epsilon$ and F^{-1} denoting the inverse function of F.
- a uniform bound on the probability that both X_1 and X_{1+m} belong to an extremal interval:

$$\frac{n}{k_n}P\big(X_1\in I_n(x,y),X_{1+m}\in I_n(x,y)\big)\le(y-x)\Big(\tilde\rho(m)+D_1\frac{k_n}{n}\Big),\tag{9}$$

  for some constant $D_1\ge 0$, a sequence $\tilde\rho(m)$, $m\in\mathbb{N}$, satisfying $\sum_{m=1}^{\infty}\tilde\rho(m)<\infty$ for all $m\in\mathbb{N}$, $0<x,y\le 1+\epsilon$ and the extremal interval, $I_n(x,y)=]F^{-1}(1-yk_n/n\big),F^{-1}(1-xk_n/n)]$.
- a limiting behavior for $\{k_n\}$

$$\lim_{n\to\infty}k_n^{1/2}\Phi(k_n/n)=0\tag{10}$$

- and, for the sake of simplicity,

$$F^{-1}(1-t)=dt^{-\gamma}(1+r(t))\ ,\ \text{with}\ |r(t)|<\Phi(t)\,.\tag{11}$$

Theorem 1.7 (Drees [**9**]). *Under the conditions (8)-(11) with $l_n=o(n/k_n)$, there exist versions of the tail empirical q.f. Q_n and a centered Gaussian process e with covariance function $\tilde c$ given by*

$$\tilde c(x,y):=x\wedge y+\sum_{m=1}^{\infty}(c_m(x,y)+c_m(y,x))\in\mathbb{R}\,,\tag{12}$$

such that

$$\sup_{t\in(0,1]}t^{\gamma+1/2}(1+|\log t|)^{-1/2}\Big|k_n^{1/2}\Big(\frac{Q_n(t)}{F^{-1}(1-k_n/n)}-t^{-\gamma}\Big)-\gamma t^{-(\gamma+1)}e(t)\Big|\to 0\tag{13}$$

in probability.

In Drees (1998a,b) it has been observed that almost every estimator $\hat{\gamma}_n$, of the tail index parameter γ, based only on the k_n+1 largest order statistics can be represented as a smooth functional T (verifying some regularity conditions) applied to the tail empirical q.f. . Hill estimator, maximum likelihood estimator, the moment estimator by Dekkers *et al.* [**3**] and Pickands' estimator [**14**] are some examples. Theorem 2.2 in Drees (2002) establishes the asymptotic normality of these estimators, more precisely, $k_n^{1/2}(\hat{\gamma}_n-\gamma) \underset{n\to\infty}{\longrightarrow} \mathcal{N}(0,\sigma^2_{T,\gamma})$ weakly with

$$\sigma^2_{T,\gamma}=\gamma^2\int_{(0,1]}\int_{(0,1]}(st)^{-(\gamma+1)}\tilde{c}(s,t)\nu_{T,\gamma}(ds)\nu_{T,\gamma}(dt), \tag{14}$$

where $\tilde{c}$ is the function defined in (12).

Hill estimator is perhaps the most popular one for heavy-tailed distributions, i.e., for $\gamma>0$ and it is defined as follows:

$$\hat{\gamma}_n^H=\frac{1}{k_n}\sum_{i=1}^{k_n}\log\frac{X_{n-i+1:n}}{X_{n-k_n:n}}. \tag{15}$$

It can be proved that it has a signed measure given by

$$\nu_{H,\gamma}(dt)=t^{\gamma}dt-\delta_1(dt), \tag{16}$$

where δ_1 is the Dirac measure with mass 1 at 1.

2. The *ARMAX* process

2.1. A stationary distribution

The *max-autoregressive* process ($ARMAX$) has emerged in EVT and became a quite popular model, specially because it has a nice treatment in what concerns extremal behavior and has very flexible marginal distributions. This kind of process can be found in literature formulated in different ways (see for instance Alpuim ([**1**]) and Canto e Castro ([**2**])). Here we consider its simplest formulation:

$$X_i=max(cX_{i-1},Z_i)\ ,\ 0<c<1, \tag{17}$$

where $\{Z_n\}$ is a sequence of i.i.d. r.v.'s with common d.f. F_Z independent from X. In Canto e Castro (1992) we can find a wide study of this process considered in a more elaborated form as it involves a threshold. There, it is showed that it has a stationary distribution, H, such that

$$H(x)=\prod_{j=0}^{\infty}F_Z(x/c^j). \tag{18}$$

In particular, H satisfies the equation

$$H(x)=F_Z(x)H(x/c). \tag{19}$$

The m-step transition probability function from x to $]-\infty, y]$, is given by,

$$Q^m(x,]-\infty, y]) := P\big(X_{n+m} \le y | X_n = x\big)$$
$$= \begin{cases} \displaystyle\prod_{j=0}^{m-1} F_Z\Big(\frac{y}{c^j}\Big) & \text{, if } x \le y/c^m \\ & \\ 0 & \text{, otherwise} \end{cases} . \tag{20}$$

It has been also proved that this markovian process is regenerative and aperiodic. This implies that it is *strong mixing*, then it is also *β-mixing* and condition $\mathrm{D}(u_n)$ holds for any real sequence $\{u_n\}$. Condition $\mathrm{D}''(u_n)$ is verified for normalized levels u_n. The existence of extremal index (θ) is stated if H is in the domain of attraction of a Weibull, a Gumbel or a Fréchet distribution with $\theta = 1$ in the first two cases and $\theta = 1 - c^{1/\gamma}$ in the last one.

Proposition 2.1. *Let $\{X_i\}$ be an* ARMAX *process as defined in (17) such that $F_Z \in \mathcal{D}(G_\gamma)$ for some $\gamma > 0$. Let H be the common d.f. of $\{X_i\}$. Then H is a non-degenerate d.f. and $H \in \mathcal{D}(G_\gamma)$.*

Proof. Recall that our assumption is equivalent to

$$1 - F_Z(x) \sim x^{-1/\gamma} L_Z(x) \text{ , as } x \to \infty, \tag{21}$$

for some slowly varying function L_Z.
Taking logarithms in (18), we can write the following approximation,

$$1 - H(x) \sim \sum_{j=0}^{\infty} (1 - F_Z(x/c^j)) \text{ , as } x \to \infty$$

and applying (21), then

$$1 - H(x) \sim x^{-1/\gamma} \Bigg(L_Z(x) + \sum_{j=1}^{\infty} c^{j/\gamma} L_Z(x/c^j) \Bigg) \text{ , as } x \to \infty .$$

The series above is convergent since, by the D'Alembert criterion, the limit

$$\lim_{j\to\infty} \frac{c^{(j+1)/\gamma} L_Z(x/c^{j+1})}{c^{j/\gamma} L_Z(x/c^j)} = \lim_{j\to\infty} \frac{c^{1/\gamma} L_Z(\frac{1}{c}\frac{x}{c^j})}{L_Z(\frac{x}{c^j})} = c^{1/\gamma} < 1 .$$

Let S be its sum. Then, we have,

$$1 - H(x) \sim x^{-1/\gamma} L_H(x), \text{ as } x \to \infty \tag{22}$$

and since $L_H(x) = L_Z(x) + S$ is also a slow varying function, $H \in \mathcal{D}(G_\gamma)$. $\square$

2.2. Tail dependence index computation for *ARMAX* processes

In this section we calculate Ledford and Tawn coefficient of tail dependence, η, of the bivariate random vector, (X_1, X_{1+m}), of an *ARMAX* process.

Proposition 2.2. *Let $\{X_i\}$ be an* ARMAX *process that fulfils the conditions of Proposition 2.1. The random vector (X_1, X_{1+m}) has unit tail dependence index.*

Proof. By (22), the tail quantile function H^{-1} can be written, as $t \downarrow 0$,

$$H^{-1}\Big(1-tx\Big) \sim \Big(tx\Big)^{-\gamma} L_H\big((xt)^{-\gamma}\big)^{\gamma}. \tag{23}$$

Assuming $U = 1 - H(X_1)$ and $V = 1 - H(X_{1+m})$ in (7), we have,

$$\frac{P\big(U < tx, V < ty\big)}{P\big(U < t, V < t\big)} = \frac{P\big(X_1 > H^{-1}(1-tx), X_{1+m} > H^{-1}(1-ty)\big)}{P\big(X_1 > H^{-1}(1-t), X_{1+m} > H^{-1}(1-t)\big)}. \tag{24}$$

Developing the numerator's expression, considering the transition probability function in (20), then,

$$\begin{aligned} & P\big(X_1 > H^{-1}(1-tx), X_{1+m} > H^{-1}(1-ty)\big) \\ = \; & \int_{H^{-1}(1-tx)}^{\infty} \big(1 - Q^m(u,]-\infty, H^{-1}(1-ty)])\big) H(du) \\ = \; & tx - \int_{H^{-1}(1-tx)}^{\frac{H^{-1}(1-ty)}{c^m}} \prod_{j=0}^{m-1} F_Z\big(H^{-1}(1-ty)/c^j\big) H(du). \end{aligned}$$

The last integral is non-null only if $x > yc^{m/\gamma}$ and so, if we apply (19) and (22), we have that,

$$\begin{aligned} & P\big(X_1 > H^{-1}(1-tx), X_{1+m} > H^{-1}(1-ty)\big) \\ = \; & tx - H(H^{-1}(1-ty)) - \tfrac{H(H^{-1}(1-tx))H(H^{-1}(1-ty))}{H\left(\frac{H^{-1}(1-ty)}{c^m}\right)} \\ \sim \; & tx - (1-ty) - \tfrac{(1-tx)(1-ty)}{1-tyc^m} \\ \sim \; & tyc^m. \end{aligned}$$

Therefore, we conclude that,

$$P\big(X_1 > H^{-1}(1-tx), X_{1+m} > H^{-1}(1-ty)\big) \sim \begin{cases} tx \text{ , if } 0 < x \le yc^{m/\gamma} \\ \\ tyc^m \text{ , if } yc^{m/\gamma} < x \le 1+\epsilon \end{cases} . \tag{25}$$

Note that, if we take $x = y = 1$, we obtain an approach to the denominator in (24) as well, given by,

$$P\big(X_1 > H^{-1}(1-t), X_{1+m} > H^{-1}(1-t)\big) \sim t \tag{26}$$

as $t \downarrow 0$. Substituting (25) and (26) in (24), the function $c(x, y)$ given in (7) becomes,

$$c(x,y) = \lim_{t \downarrow 0} \frac{P\big(U < tx, V < ty\big)}{P\big(U < t, V < t\big)} \sim \begin{cases} x \text{ , if } 0 < x \le yc^{m/\gamma} \\ \\ yc^m \text{ , if } yc^{m/\gamma} < x \le 1+\epsilon \end{cases} ,$$

which is an homogeneous function of order 1. Hence $\eta = 1$ and we conclude that two tail observations from an $ARMAX$ process, that are distant $m+1$ time instants from each other, are asymptotically dependent for any given value of the model constant c in (17). □

2.3. The tail empirical quantile function for $ARMAX$ processes

We are going to apply Drees result presented in Section 1.3.

Proposition 2.3. *Let* $\{X_i\}$ *be an* ARMAX *process that fulfils the conditions of Proposition 2.1. Then (8) and (9) both hold.*

Proof.

The limit in (8) is immediate if we replace t by k_n/n in (25) with $n \to \infty$. Hence, we have,

$$c_m(x,y) = \begin{cases} x & , 0 < x \le y c^{m/\gamma} \\ y c^{m/\gamma} & , y c^{m/\gamma} < x \le 1+\epsilon \end{cases} . \tag{27}$$

Now let us see that the second condition (9) also holds. Considering $I_n(x,y) =]H^{-1}(1-yk_n/n), H^{-1}(1-xk_n/n)]$, we have,

$$\begin{aligned} & \tfrac{n}{k_n} P\Big(X_1 \in I_n(x,y), X_{1+m} \in I_n(x,y)\Big) \\ \le\ & \tfrac{n}{k_n} P\Big(X_1 \in I_n(x,y), X_{1+m} > H^{-1}(1-yk_n/n)\Big) \\ \le\ & \tfrac{n}{k_n} P\Big(X_1 \in I_n(x,y), X_1 > c^{-m} H^{-1}(1-yk_n/n)\Big) \\ & + \tfrac{n}{k_n} P\Big(X_1 \in I_n(x,y), \max_{k=2,\dots,1+m} \big(c^{m-k+1} Z_k\big) > H^{-1}(1-yk_n/n)\Big) \\ =\ & \tfrac{n}{k_n}\Big\{1 - \tfrac{k_n}{n}x - H\Big(\tfrac{H^{-1}(1-\frac{k_n}{n}y)}{c^m}\Big) + \tfrac{k_n}{n}(y-x)\Big[1 - \prod_{k=2}^{m+1} F_Z\Big(\frac{H^{-1}(1-\frac{k_n}{n}y)}{c^{m-k+1}}\Big)\Big]\Big\}, \end{aligned}$$

where the last step is due to the independence of the r.v.'s Z_n and the independence between r.v.'s X and Z. Applying the stationarity deduction for the distribution H in (18) and some more calculations, then,

$$\begin{aligned} & \tfrac{n}{k_n} P\Big(X_1 \in I_n(x,y), X_{1+m} \in I_n(x,y)\Big) \\ \le\ & \tfrac{n}{k_n}\Big\{\tfrac{k_n}{n} y c^{m/\gamma} - \tfrac{k_n}{n}x + H\Big(\tfrac{H^{-1}(1-\frac{k_n}{n}y)}{c^m}\Big) + \\ & \quad + \tfrac{k_n}{n}(y-x)\Big[1 - \prod_{j=0}^{\infty} F_Z\Big(\frac{H^{-1}(1-\frac{k_n}{n}y)}{c^j}\Big)\Big]\Big\} \\ \le\ & (y-x)\Big(c^{m/\gamma} + \tfrac{k_n}{n}y\Big). \end{aligned}$$

Because $y \in (0, 1+\epsilon]$, if we take $D_1 = 1+\epsilon$ and $\tilde{\rho}(m) = c^{m/\gamma}$, then condition in (9) holds, since $\sum_{m=1}^{\infty} c^{m/\gamma} < \infty$. □

Proposition 2.4. *Let* $\{X_i\}$ *be the an* ARMAX *process that fulfils the conditions of Proposition 2.1 and* $\{k_n\}$ *an intermediate sequence. Then, for the tail empirical quantile function*

$Q_n(t)$, (13) is valid with

$$\tilde{c}(x,y) = \min(x,y) + \sum_{m=1}^{p-1}(c_m(x,y)+c_m(y,x)) + (x+y)\frac{c^{p/\gamma}}{(1-c^{1/\gamma})}\ , \tag{28}$$

where $p \equiv p_{x,y} = [\max\{\gamma\ln(x/y)/\ln c, \gamma\ln(y/x)/\ln c\}]+1$.

Considering $\hat{\gamma}^H$ the tail index Hill estimator, then $k_n^{1/2}(\hat{\gamma}^H - \gamma) \longrightarrow \mathcal{N}(0,\sigma^2_{H,\gamma})$ weakly with

$$\sigma^2_{H,\gamma} = \gamma^2 \int_{(0,1]}\int_{(0,1]} (st)^{-(\gamma+1)}\tilde{c}(s,t)\nu_{H,\gamma}(ds)\nu_{H,\gamma}(dt), \tag{29}$$

where the signed measure, $\nu_{H,\gamma}$, is defined in (16.

Proof. Since $\{X_i\}$ is *β-mixing*, (13) and the asymptotic normality of $\hat{\gamma}^H$ follow immediately from Proposition 2.3.
The covariance function of the centered Gaussian process, $c(x,y)$, is given by (12), where

$$c_m(x,y)+c_m(y,x) = \begin{cases} x(1+c^{m/\gamma}) & , 0 < x \le yc^{m/\gamma} \\ (y+x)c^{m/\gamma} & , yc^{m/\gamma} < x \le yc^{-m/\gamma} \\ y(1+c^{m/\gamma}) & , yc^{-m/\gamma} < x \le 1+\epsilon \end{cases} . \tag{30}$$

Note that $c^{m/\gamma} \to 0$ and $c^{-m/\gamma} \to \infty$, as $m \to \infty$. If we fix both x and y, it exists some order $p \in \mathbb{N}$ such that, for all $m \ge p$, the condition $yc^{m/\gamma} < x \le yc^{-m/\gamma}$ is always true and so

$$\sum_{m=1}^{\infty}(c_m(x,y)+c_m(y,x)) = \sum_{m=1}^{p-1}(c_m(x,y)+c_m(y,x)) + \sum_{m=p}^{\infty}(x+y)c^{m/\gamma}.$$

As the second member series is geometric with ratio $c < 1$, then (28) follows, taking an order $p = p_{x,y}$ as stated above. □

3. The *ARMAX-p* process

The *power max-autoregressive* (*ARMAX-p*) process is generated in a quite similar way as the *ARMAX*: it considers a power transformation instead of a product. Therefore, considering an i.i.d. sequence $\{Z_n\}$ with positive support and common d.f. F_Z, we say that $\{X_n\}$ is a *ARMAX-p* if

$$X_i = max(X_{i-1}^c, Z_i)\ ,\ 0 < c < 1 \tag{31}$$

with $\{Z_n\}$ also independent from $\{X_n\}$.
This markovian process was constructed so that its coefficient of tail dependence can change with the values of c. Before going on this assertion, let us show some features about the *ARMAX-p* process. It is obvious that *ARMAX-p* would be useless for our purposes if there was no chance of a stationary performance. Next section is devoted to the existence of a non-degenerate stationary distribution.

3.1. The existence of a stationary and non-degenerate distribution

Proposition 3.1. *A* ARMAX-p *process* $\{X_i\}$ *admits a stationary distribution* H_p*, such that*

$$H_p(x) = \prod_{j=0}^{\infty} F_Z\big(x^{1/c^j}\big). \tag{32}$$

Proof. Suppose that H_n is the d.f. of X_n. Then

$$(H_p)_n(x) = P(X_n \leq x) = P(X_{n-1}^c \leq x, Z_n \leq x) = (H_p)_{n-1}(x) x^{1/c} F_Z(x)$$

So, we are looking for a solution of the equation

$$H_p(x) = H_p(x^{1/c}) F_Z(x), \tag{33}$$

i.e., a d.f. $H_p(x)$ that satisfies the upper relation for all x. Calculating recursively, we have,

$$H_p(x) = H_p(x^{1/c}) F_Z(x) = H_p(x^{1/c^2}) F_Z(x^{1/c}) F_Z(x) = ... =$$

$$= H_p(x^{1/c^n}) F_Z(x^{1/c^{n-1}}) ... F_Z(x^{1/c}) F_Z(x),$$

which must hold for all $n \in \mathbb{N}$. Letting n go to infinity, we obtain (32) since H_p is a d.f. and so, $H_p(\infty) = 1$. □

We must also assure that H_p is non-degenerate. This holds under some mild requirements on the tail of the marginal distributions.

Proposition 3.2. *Let* $\{X_i\}$ *be a* ARMAX-p *process with common d.f.* H_p*. If* $F_Z \in \mathcal{D}(G_\gamma)$*, for some* $\gamma > 0$*, then* H_p *is a non-degenerate distribution function such that* $H_p \in \mathcal{D}(G_\gamma)$*.*

Proof. If we take logarithms in (32), our assumption leads us to

$$1 - H_p(x) \quad \sim \quad \sum_{j=0}^{\infty} x^{-1/(\gamma c^j)} L_Z\big(x^{1/c^j}\big) \text{ , as } x \to \infty \, ,$$

with L_Z a slowly varying function. In order to get an approach to the above sum, we use the integral criterium and state an approach to

$$\int_0^\infty x^{-1/(\gamma c^y)} L_Z(x^{1/c^y}) dy \, ,$$

which, making the replacement $u = x^{1/c^y}$, becomes

$$\int_x^\infty u^{-1/\gamma} L_Z(u) \Big(\frac{-1}{u \log u \log c} \Big) du = \int_x^\infty u^{-1/\gamma - 1} L^*(u) du \, ,$$

with L^* a slowly varying function. By Karamata's Theorem, we have,

$$\int_x^\infty u^{-1/\gamma - 1} L^*(u) du \sim x^{-1/\gamma} L_{H_p}(x) \, ,$$

where $L_{H_p}(x) = -1/\gamma L^*(x)$ is a slowly varying function.

Therefore, we have,

$$1 - H_p(x) \sim x^{-1/\gamma} L_{H_p}(x) \text{ , as } x \to \infty \tag{34}$$

and hence, H_p is a non-degenerate d.f., such that, $H_p \in \mathcal{D}(G_\gamma)$. □

3.2. The dependence structure

Let us start by computing the transition probability and the density function of the markovian process *ARMAX-p*. The m-step transition probability function from x to $]-\infty, y]$, becomes,

$$Q^m(x,]-\infty, y]) := P\big(X_{n+m} \leq y | X_n = x\big) = \begin{cases} \prod_{j=0}^{m-1} F_Z(y^{1/c^j}) & \text{, if } x \leq y^{1/c^m} \\ 0 & \text{, otherwise} \end{cases} . \tag{35}$$

In order to show that *ARMAX-p* is *strong-mixing*, we are going to use the general theory of the markovian processes which states that regeneration jointly with aperiodicity is a sufficient requirement.

Consider $\{X_n\}_{n\in\mathbb{N}}$ a markovian and stationary process with common d.f. F. A set R is said to be *recurrent* if $F(R) = \int_R F(dx) > 0$. In addition, if for some $m > 0$, $\epsilon \in (0,1)$ and some distribution λ, we have

$$Q^m(x, B) \geq \epsilon\lambda(B),\ x \in R \tag{36}$$

for all $B \in \mathcal{B}(\mathbb{R})$ ($\mathcal{B}(\mathbb{R})$ is the Borel σ-field), then R is called a *regeneration* set. We say that $\{X_i\}$ is a *regenerative* process if it has a regeneration set.

We say that $\{X_i\}$ is an *aperiodic* process if for any regeneration set R and for any event B,

$$Q^{m+1}(x, B) \geq \epsilon_1\lambda(B) \text{ and } Q^m(x, B) \geq \epsilon_2\lambda(B)\ , \forall x \in R, \tag{37}$$

for some $m \in \mathbb{N}$ and $\epsilon_1, \epsilon_2 \in (0,1)$.

Proposition 3.3. *Let $\{X_i\}$ be an* ARMAX-*p process that fulfils the conditions of Proposition 3.2. Then $\{X_i\}$ is regenerative and aperiodic.*

Proof. Let $R = [r, +\infty[$ be a proper subset of the support of F_Z. Note that,

$$H_p(R) = 1 - \prod_{j=0}^{\infty} F_Z(r^{1/c^j}) = 1 - F_Z(r)\prod_{j=1}^{\infty} F_Z(r^{1/c^j}) > 0,$$

because $0 < F_Z(r) < 1$ (r belongs to the support of F_Z) and so R is recurrent. Denote by μ the Lebesgue measure in $\mathcal{B}(\mathbb{R})$ and let $S \subset R$ be a compact such that $\mu(S) < 1$. There exists some $0 < \delta < 1$ such that $f_Z(y) \geq \delta$, $\forall y \in S$. Hence, for $x \in R$,

$$Q(x, B) \geq \int_{B\cap S} f_Z(y)dy \geq \delta\mu(B \cap S) = \delta\mu(S)\lambda(B)$$

with $\lambda(.) = \mu(. \cap S)/\mu(S)$ the uniform distribution in S. The condition (36) is verified taking $m = 1$ and $\epsilon = \delta\mu(S)$ (note that $0 < \delta\mu(S) < 1$).
To prove aperiodicity, and since R is a regenerative set, it is sufficient to see that,

$$\begin{aligned} Q^2(x,]-\infty, y]) &= \int P(X_{n+2} \leq y | X_{n+1} = z) Q(x, dz) \\ &\geq \int_S P(X_{n+2} \leq y | X_{n+1} = z) f_Z(z) dz \\ &\geq \int_S Q(z,]-\infty, y]) dz \\ &\geq \delta\mu(S)\epsilon\lambda(]-\infty, y]) \end{aligned}$$

The condition (37) holds taking $\epsilon_1 = \delta\mu(S)\epsilon$ and $\epsilon_2 = \epsilon$. $\square$

By the previous proposition we conclude that *ARMAX-p* is *strong mixing*, hence *β-mixing* and so, condition $\mathrm{D}(u_n)$ holds for any real sequence $\{u_n\}$.

Proposition 3.4. *If $\{X_i\}$ is a* ARMAX-p *process that fulfils the conditions of Proposition 3.2 and $\{u_n\}$ is a real sequence such that $1 - H_p(u_n) = O(1/n)$, then condition $D''(u_n)$ holds.*

Proof. For $j \geq 2$, we have,

$$\begin{aligned} & P\big(X_1 > u_n, X_j \leq u_n < X_{j+1}\big) \\ = & \int_{u_n}^{\infty} P\big(X_j \leq u_n < X_{j+1} | X_1 = y\big) H_p(dy) \\ = & \int_{u_n}^{\infty} \int_{-\infty}^{u_n} P\big(X_{j+1} > u_n | X_j = z\big) Q^{j-1}(y, dz) H_p(dy) \\ = & \int_{u_n}^{\infty} \int_{-\infty}^{u_n} Q(z,]u_n, \infty]) Q^{j-1}(y, dz) H_p(dy) \end{aligned} \tag{38}$$

Since $z < u_n$ then, by (35), $Q(z,]-\infty, u_n]) = F_Z(u_n)$. A substitution in the last integral gives

$$\begin{aligned} & \int_{u_n}^{\infty} \int_{-\infty}^{u_n} (1 - F_Z(u_n)) Q^{j-1}(y, dz) H_p(dy) \\ = & \ (1 - F_Z(u_n)) \int_{u_n}^{\infty} Q^{j-1}(y,]-\infty, u_n]) H_p(dy) \\ & \leq (1 - F_Z(u_n))(1 - H_p(u_n)), \end{aligned} \tag{39}$$

Note that $1 - F_Z(x) = 1 - H_p(x)/H_p(x^{1/c}) \leq 1 - H_p(x)$. Hence, condition $\mathrm{D}''(u_n)$ holds since, by (38) and (39),

$$
\begin{aligned}
n \sum_{j=2}^{r_n - 1} P\big(X_1 > u_n, X_j \leq u_n < X_{j+1}\big) &\leq n \sum_{j=2}^{r_n-1} (1 - F_Z(u_n))(1 - H_p(u_n)) \\
&\leq n \tfrac{n}{k_n} (1 - F_Z(u_n))(1 - H_p(u_n)) \\
&= O(1/k_n) \to 0 \text{ , as } n \to \infty. \ \square
\end{aligned}
$$

Proposition 3.5. *Let $\{X_i\}$ be an* ARMAX-*p process that fulfils the conditions of Proposition 3.2 and $\{u_n(\tau)\}$ a real sequence such that $n(1 - H_p(u_n(\tau))) \to \tau$, as $n \to \infty$, with $\tau > 0$. Then, $\{X_i\}$ has unit extremal index.*

Proof. Condition $\mathrm{D}(u_n(\tau))$ obviously hold and, by Proposition 3.4, condition $\mathrm{D}''(u_n(\tau))$ holds as well. Therefore, we can apply Theorem 1.6:

$$
\begin{aligned}
\theta &= \lim_{n\to\infty} P\big(X_2 \leq u_n(\tau) | X_1 > u_n(\tau)\big) \\
&= \lim_{n\to\infty} \frac{P\big(X_2 \leq u_n(\tau), X_1 > u_n(\tau)\big)}{P\big(X_1 > u_n(\tau)\big)} \\
&= \lim_{n\to\infty} \frac{\int_{u_n(\tau)}^{\infty} P\big(X_2 \leq u_n(\tau) | X_1 = y\big) H_p(dy)}{1 - H_p(u_n(\tau))} \\
&= \lim_{n\to\infty} \frac{\int_{u_n(\tau)}^{u_n(\tau)^{1/c}} F_Z(u_n(\tau)) H_p(dy)}{1 - H_p(u_n(\tau))} \\
&= \lim_{n\to\infty} F_Z(u_n(\tau)) \frac{H_p(u_n(\tau)^{1/c}) - H_p(u_n(\tau))}{1 - H_p(u_n(\tau))} \\
&= \lim_{n\to\infty} \left[1 - \frac{1 - H_p(u_n(\tau)^{1/c})}{1 - H_p(u_n(\tau))}\right].
\end{aligned}
$$

The result follows by (34), since

$$
\begin{aligned}
\lim_{n\to\infty} \left[1 - \frac{1 - H_p(u_n(\tau)^{1/c})}{1 - H_p(u_n(\tau))}\right] &= \lim_{n\to\infty} \left[1 - \frac{u_n(\tau)^{-\gamma/c} L_{H_p}(u_n(\tau)^{1/c})}{u_n(\tau)^{-\gamma} L_{H_p}(u_n(\tau))}\right] \\
&= \lim_{n\to\infty} \left[1 - u_n(\tau)^{-\gamma(1/c-1)}\right] \\
&= 1. \ \square
\end{aligned}
$$

3.3. The tail dependence index computation for *ARMAX*-pprocesses

As we mentioned in the beginning of this section, our main purpose is to determine Ledford and Tawn coefficient of tail dependence, η, presented in section 1.2, which runs along similar steps as the *ARMAX* process.

Proposition 3.6. *Let $\{X_i\}$ be an* ARMAX-p *process that fulfils the conditions of Proposition 3.2. Then, the random vector (X_1, X_{1+m}) has $\eta = 1/2$, if $c \leq (1/2)^{1/m}$ and $\eta = c^m$, otherwise.*

Proof. In order to obtain function the $c(x, y)$, we take $U = 1 - H_p(X_1)$ and $V = 1 - H_p(X_{1+m})$ and calculate the limit in (7). An analogous procedure used in the proof of Proposition 2.2 lead us to,

$$P(X_1 > H_p^{-1}(1-tx), X_{1+m} > H_p^{-1}(1-ty)) = t(x+y) - 1 + \frac{(1-tx)(1-ty)}{H_p(H_p^{-1}(1-ty)^{1/c^m})}$$

and applying (34) in the latter expression as $t \downarrow 0$,

$$\begin{aligned} P(X_1 > H_p^{-1}(1-tx), X_{1+m} > H_p^{-1}(1-ty)) &\sim t(x+y) - 1 + \frac{(1-tx)(1-ty)}{1-(ty)^{1/c^m} k^*} \\ &\sim \begin{cases} xyt^2 & , \text{if } c < (1/2)^{1/m} \\ k^*(yt)^{1/c^m} & , \text{if } c > (1/2)^{1/m} \end{cases} , \end{aligned}$$

for some positive constant k^*. If we replace x and y by 1, we also obtain, as $t \downarrow 0$,

$$\begin{aligned} P(X_1 > H_p^{-1}(1-t), X_{1+m} > H_p^{-1}(1-t)) &\sim 2t - 1 + \frac{(1-t)^2}{1-t^{1/c^m} k^*} \\ &\sim \begin{cases} t^2 & , \text{if } c < (1/2)^{1/m} \\ k^* t^{1/c^m} & , \text{if } c > (1/2)^{1/m} \end{cases} . \end{aligned}$$

Hence, the function $c(x, y)$ for the *ARMAX*-p process, is given by,

$$c(x,y) = \begin{cases} xy & , \text{if } c < (1/2)^{1/m} \\ y^{1/c^m} & , \text{if } c > (1/2)^{1/m} \end{cases}$$

and so, $\eta = 1/2$ if $c < (1/2)^{1/m}$ and $\eta = c^m$ if $c > (1/2)^{1/m}$.
If $c = (1/2)^{1/m}$, we have, respectively, $c(x, y) = xy$, $c(x, y) = y^2$ or $c(x, y) = (xy + k^* y^2)/(1 + k^*)$, whenever $L_{H_p} \to 0$, $L_{H_p} \to \infty$ or L_{H_p} in (34) converges to some positive constant. Once again, $\eta = 1/2$. Therefore, the assertion follows. □

In a future work, we will try to apply Drees result presented in subsection 1.3. The limiting functions, c_m, in (8), are computed in the next result:

Proposition 3.7. *Let $\{X_i\}$ be an* ARMAX-p *process that fulfils the conditions of Proposition 3.2. Then (8) holds with $c_m(x, y) = 0$, for all $m \in \mathbb{N}$.*

Proof. Replacing t by k_n/n in Proposition 3.6 proof and applying (8), we have, as $n \to \infty$,

$$\begin{aligned} c_m(x,y) &= \lim_{n\to\infty} \frac{n}{k_n} P\Big(X_1 > H_p^{-1}\Big(1-\frac{k_n}{n}x\Big), X_{1+m} > H_p^{-1}\Big(1-\frac{k_n}{n}y\Big)\Big) \\ &\sim \begin{cases} xy\frac{k_n}{n} & \text{, if } c < (1/2)^{1/m} \\ k^*(y\frac{k_n}{n})^{1/c^m-1} & \text{, if } c > (1/2)^{1/m} \end{cases}, \end{aligned}$$

and, for $c = (1/2)^{1/m}$,

$$c_m(x,y) \sim \begin{cases} xy\frac{k_n}{n} & \text{, if } L_{H_p}(x) \to 0 \\ y^2\frac{k_n}{n} & \text{, if } L_{H_p}(x) \to \infty \\ \frac{xy+k^*y^2}{1+k^*}\frac{k_n}{n} & \text{, if } L_{H_p}(x) \to a^*(a^*>0) \end{cases}.$$

Hence, the assertion follows. $\square$

References

[1] Alpuim, M.T. (1989). *Contribuições à teoria de extremos em sucessões dependentes.*. Ph. D. Thesis, FCUL.

[2] Canto e Castro, L. (1992). *Sobre a Teoria Assintótica de Extremos.* Ph. D. Thesis, FCUL.

[3] Dekkers, A.L.M., Einmahl, J.H.J. and de Haan, L. (1989). A moment estimator for the index of an extreme value distribution. *Ann. Statist.* **17**, 1833-1855.

[4] Draisma, G. (2001). *Duration of extremes at sea.* Em *Parametric and semi-parametric methods in E.V.T.*, p. 137-143. Ph. D. Thesis, Erasmus. University.

[5] Draisma,G., Drees, H., Ferreira, A. and de Haan, L. (2004). Bivariate tail estimation: dependence in asymptotic independence. *Bernoulli* **10**, 251-280.

[6] Drees, H. (1998a). On smooth statistical tail functionals. *Scand. J. Statist.* **25**, 187-210.

[7] Drees, H. (1998b). A general class of estimators of the extreme value index. *J. Statist. Plann. Inference* **66**, 95-112.

[8] Drees, H. (2000). Weighted approximations of tail processes for β-mixing random variables.

[9] Drees, H. (2002). Extreme quantile estimation for dependent data with applications to finance.

[10] Leadbetter, M.R., Lindgren, G. and Rootzén, H. (1983.). *Extremes and Related Properties of Random Sequences and Processes.* Springer-Verlag, New-York, Heidelberg-Berlin.

[11] Leadbetter, M.R. and Nandagopalan, S. (1989.). *On Exceedance Point Processes for Stationary Sequences under Mild Oscillation Restrictions.* Em *Extreme Value Theory.* (J. Hüsler and R.-D. Reiss, eds.), p. 69-80. Springer-Verlag.

[12] Ledford, A. and Tawn, J.A. (1996). Statistics for near independence in multivariate extreme values. *Biometrika* **83**, 169-187.

[13] Ledford, A. and Tawn, J.A. (1997). Modelling Dependence within joint tail regions. *J. Royal Statist. Soc.Ser. B* **59** 475-499.

[14] Pickands III, J. (1975) Statistical inference using extreme order statistics. *Ann. Statist.* **3**, 74-77.

[15] Watson, G.S. (1954.). Extreme Values for Approximate Independence of Largest and Smallest Order Statistics. *J.Amer.Statist.Assoc.*, Vol.65, p. 860-863.