Contributions for the study of high levels that persist over a fixed period of time

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Abstract: This work was motivated by a study presented in Draisma [**2**], in which it is analyzed the extremal behaviour of a series $\{X_i\}$ of daily extreme sea water levels. In its approach, Draisma defines a new series, $\{Y_i\}$, as the minimum of a fixed number of successive high tide water levels. This is obviously a dependent series and we analyze its dependence feature, whether $\{X_i\}$ is an i.i.d. sequence or has some specific dependence structure. We also study its tail behavior in case $\{X_i\}$ is an i.i.d. sequence and assuming that the common distribution of X_i is in the domain of attraction of a generalized extreme value distribution.

Keywords: Statistic of extremes, Tail index, Extremal index, Stationary sequence.

1. Introduction

Extreme Value Theory (EVT) has a recognized importance in nowadays society as it is necessary to predict the probability of occurrence of many adverse situations such as floods, wind storms or crash markets. The extremal models for dependent or independent sequences, allow good estimates of the probability of observe great (small) values of the variable that we are interested in. Nevertheless, it may happened that the adverse situation comes from the duration of exceedances above (bellow) given thresholds in a fixed period of time. In Draisma [**2**], for instance, the main problem is the successive high tide water levels registered on some places of Holland's coast. Its persistence in time can contribute for the sand dunes damage and may give rise to devastating floods. More formally, given a time series of water levels, $\{X_1, ..., X_n\}$, we are interested on the tail behaviour of $Y_i = \min(X_i, X_{i+1}, ..., X_{i+s})$, which is a level that persists for $s+1$ periods. Our study focus on this new sequence, $\{Y_i\}$.

We start by presenting some basic results of the classical extreme value theory (EVT) and some general local and asymptotic dependence conditions that will be used in our study. Section 3 contains the whole study and is divided in two parts. In the first one (subsection3.1), $\{X_i\}$ is an i.i.d. sequence and we prove that the dependence conditions $\mathrm{D}(u_n)$ and $\mathrm{D}'(u_n)$ (Leadbetter [**3**]) both hold for $\{Y_i\}$. We also show that, if the distribution function (d.f.) of X_i is in the domain of attraction of a non-degenerate generalized extreme value distribution (GEV) then Y_i is in the same domain of attraction having a tail index that depends both on the tail index of X_i and the fixed period of time, $s+1$. In subsection 3.2, $\{X_i\}$ is a stationary sequence with some specific weak dependence structure. We shall see that these conditions hold for $\{Y_i\}$ whenever they hold for $\{X_i\}$. In what concerns the tail behaviour of $\{Y_i\}$, it depends on the d.f. of X_i as well as on its dependence feature.

2. Preliminary Results

We start with the classical central result in extreme value theory:

Theorem 2.1. *Let $\hat{M}_n = \max(\hat{X}_1, ..., \hat{X}_n)$, where $\hat{X}_1, \hat{X}_2, ...$ are i.i.d. random variables. If for some constants $a_n > 0$, b_n, we have*

$$P(\hat{M}_n \leq a_n x + b_n) \underset{n\to\infty}{\longrightarrow} G(x) \tag{1}$$

for some non-degenerate G, then G is one of the three extreme value types distributions

$$\text{Type I (Gumbel)}: \quad G(x) = exp(-e^{-x}) \ , \quad -\infty < x < \infty;$$

$$\text{Type II (Fréchet)}: \quad G(x) = \begin{cases} 0 & , x \leq 0 \\ exp(-x^{-\alpha}) \ , \text{for some } \alpha > 0 & , x > 0; \end{cases}$$

$$\text{Type III (Weibull)}: \quad G(x) = \begin{cases} exp(-(-x)^{\alpha}) \ , \text{for some } \alpha > 0 & , x \leq 0 \\ 1 & , x > 0. \end{cases}$$

Conversely, each d.f. G of extreme value type may appear as a limit in (1). G can be uniquely represented in the following parametric way, known as Generalized Extreme Value function (GEV),

$$G_\gamma(x) = exp(-(1+\gamma x)^{-1/\gamma}), \ 1+\gamma x > 0, \ \gamma \in \mathbb{R},$$

which is interpreted as $exp(-e^{-x})$ if $\gamma = 0$. The parameter γ is the tail index with $\gamma > 0$ corresponding to the Fréchet case and $\gamma < 0$ corresponding to the Weibull case .

We say that F is in the domain of attraction of G_γ, denoted by $F \in \mathcal{D}(G_\gamma)$, if the limit (1) is verified for some real constants $a_n > 0$ e b_n.

Another important result follows:

Theorem 2.2. *Let $0 \leq \tau \leq \infty$ and suppose that $\{u_n\}$ is sequence of real numbers such that*

$$n(1 - F(u_n)) \to \tau \text{ as } n \to \infty. \tag{2}$$

Then,

$$P(\hat{M}_n \leq u_n) \to e^{-\tau} \text{ as } n \to \infty. \tag{3}$$

Conversely, if (3) holds for some τ, $0 \leq \tau \leq \infty$, then so does (2).

Remark 2.1 When (2) holds for some $\tau \geq 0$, u_n is said to be a normalized level and it is denoted by $u_n(\tau)$.

Now we present the definitions of the weak dependence conditions that will be considered. They have in common the fact that dependence vanishes as r.v.'s become more distant in time.

Let us start by the simplest type of dependence considered here (Watson [**5**]):

Definition 2.1. *The sequence $\{X_i\}$ is m-dependent, if the r.v.'s X_i e X_j are independent whenever $|i-j|>m$.*

It follows the so-called condition D(u_n) (Leadbetter, *et al.* [**3**]). Let $\{X_i\}$ be a stationary sequence, F the common d.f. of the r.v.'s X_i and $\{u_n\}$ a given real sequence. For brevity, consider the notation:

$$P(X_{i_1}\leq u_n,...,X_{i_m}\leq u_n)=F_{i_1,...,i_m}(u_n,...,u_n)=F_{i_1,...,i_m}(u_n),$$

and

$$P(X_{i_1}>u_n,...,X_{i_m}>u_n)=P((X_{i_1},...,X_{i_m})>u_n)=\overline{F}^X_{i_1,...,i_m}(u_n),$$

for any integers $i_1<...<i_m$.

Definition 2.2. *Condition D(u_n) will be said to hold for $\{X_i\}$, if for any integers*

$$1\leq i_1<...<i_p<j_1<...<j_{p'}\leq n \quad \textit{with } j_1-i_p\geq \ell,$$

then

$$|F^X_{i_1,...,i_p,j_1,...,j_{p'}}(u_n)-F^X_{i_1,...,i_p}(u_n)F^X_{j_1,...,j_{p'}}(u_n)|\leq \alpha_{n,\ell}$$

where $\alpha_{n,\ell_n}\underset{n\to\infty}{\longrightarrow}0$ for some sequence $\ell_n=o(n)$.

In a D-dependent context, some limiting results of classical EVT for i.i.d. sequences also hold. Extremal Types Theorem and Theorem 2.2 are some examples as we shall see.

Theorem 2.3. *Let $M_n=\max(X_1,...,X_n)$ and $\{a_n>0\}$, $\{b_n\in\mathbb{R}\}$ real sequences such that $P\big(M_n\leq a_nx+b_n\big)$ converges to a non-degenerate function $G(x)$. If $D(a_nx+b_n)$ holds for each real x, then $G(x)$ is a GEV distribution.*

Non-independency may give rise to a new parameter, θ, known as *extremal index*:

Theorem 2.4. *If for any given positive constant τ there is a normalized levels sequence $u_n(\tau)$ such that $D(u_n(\tau))$ holds and for some τ_*, $P\big(M_n\leq u_n(\tau_*)\big)$ converges then*

$$P\big(M_n\leq u_n(\tau)\big)\underset{n\to\infty}{\longrightarrow}e^{-\theta\tau}\ , \textit{ for all } \tau>0$$

with $\theta\in[0,1]$ constant.

Definition 2.3 (Leadbetter *et al.* [**3**])**.** *$\{X_i\}$ has extremal index θ ($0\leq\theta\leq1$) if for each $\tau>0$ there is a normalized levels sequence $u_n(\tau)$ such that*

$$P\big(M_n\leq u_n(\tau)\big)\underset{n\to\infty}{\longrightarrow}e^{-\theta\tau},$$

with θ independent from τ.

Hence, i.i.d. sequences whose normalized maximum converges have unit extremal index.

Another local dependence condition is considered in Leadbetter *et al.* [**3**] which bounds the probability of more than one exceedance of u_n, on a time-interval of $[n/k]$ integers as $k\to\infty$.

Definition 2.4. *Condition $D'(u_n)$ will be said to hold for $\{X_i\}$ if*

$$\limsup_{n\to\infty} n \sum_{j=2}^{[n/k]} P\big(X_1 > u_n, X_j > u_n\big) \underset{k\to\infty}{\longrightarrow} 0.$$

If condition $\mathrm{D}'(u_n)$ holds, the exceedances of level u_n tend to come out isolated, similar to an i.i.d. behaviour. Leadbetter *et al.* [**3**] consider a generalization of Theorem 2.2 in order to apply to stationary sequences under $\mathrm{D}(u_n)$ and $\mathrm{D}'(u_n)$.

Theorem 2.5. *Let $\{u_n\}$ be a real sequence such that $D(u_n)$ e $D'(u_n)$ hold for $\{X_i\}$ and $0 \le \tau < \infty$. Then*

$$P\big(M_n \le u_n\big) \to e^{-\tau} \text{ if and only if } n(1 - F(u_n)) \to \tau \text{ , as } n \to \infty.$$

A sequence verifying the conditions of the latest theorem has an unit extremal index.

If condition $\mathrm{D}'(u_n)$ doesn't hold, then the exceedances of u_n tend to cluster. For such sequences, Leadbetter e Nandagopalan [**4**] stated another local dependence condition, $\mathrm{D}''(u_n)$, weaker than $\mathrm{D}'(u_n)$, that restricts rapid oscillations near high levels.

Definition 2.5. *Condition $D''(u_n)$ will be said to hold for $\{X_i\}$ if condition $D(u_n)$ also holds and k_n are integers such that*

$$k_n \underset{n\to\infty}{\longrightarrow} \infty \text{ , } k_n\alpha_{n,l_n} \underset{n\to\infty}{\longrightarrow} 0 \text{ , } k_n l_n/n \underset{n\to\infty}{\longrightarrow} 0 \text{ , } k_n(1 - F(u_n)) \underset{n\to\infty}{\longrightarrow} 0 \tag{4}$$

and

$$\lim_{n\to\infty} n \sum_{j=2}^{[n/k_n]-1} P\big(X_1 > u_n, X_j \le u_n < X_{j+1}\big) = 0.$$

Conditions D and D'' may be of great help in what concerns extremal index computation, especially if the transition probability function is known.

Theorem 2.6. *Let $\{u_n(\tau)\}$ be a sequence of normalized levels and assume that $D(u_n(\tau))$ and $D''(u_n(\tau))$ hold for each $\tau > 0$. If*

$$\lim_{n\to\infty} P\big(X_2 \le u_n(\tau) | X_1 > u_n(\tau)\big) = \theta \text{ ,}$$

for some $\tau > 0$, then convergence to θ occurs for all $\tau > 0$ and $\{X_i\}$ has extremal index θ.

3. High levels persisting in time

As we have already mentioned, we have an high levels sequence $\{X_i\}$ and we intend to analyze the distribution of the maximum levels that persist in a previous fixed amount of time. More precisely, we continue with Draisma [**2**] approach and consider the statistics

$$Y_i = \min(X_i, X_{i+1}, ..., X_{i+s}),$$

corresponding to high levels that persist throghout $s + 1$ time instants and we study the dependence structure and the extremal behaviour of $\{Y_i\}$.

3.1. $\{X_i\}$ i.i.d.

If $\{X_i\}$ is an i.i.d. sequence, $\{Y_i\}$ is obviously a stationary and $(s+1)$-dependent sequence. Therefore, condition $\mathrm{D}(u_n)$ holds for all $\{u_n\}$. Lets see that $\mathrm{D}'(u_n)$ also holds for a certain levels choice $\{u_n\}$.
Our assumptions lead us to the following development:

$$\begin{aligned}
&\limsup_{n\to\infty} n\sum_{i=2}^{[n/k]} P\big(Y_1>u_n, Y_i>u_n\big) = \\
=\ &\limsup_{n\to\infty} n\sum_{i=2}^{[n/k]} P\big(X_1>u_n,...,X_{1+s}>u_n, X_i>u_n,...,X_{i+s}>u_n\big) = \qquad (5)\\
=\ &\limsup_{n\to\infty} n\sum_{i=2}^{s+1}(1-F(u_n))^{i+s} + n\sum_{i=s+2}^{[n/k]}(1-F(u_n))^{2+2s}.
\end{aligned}$$

Taking $\{u_n\}$ such that

$$n^{1/(s+1)}(1-F(u_n)) \to \tau \text{ , as } n\to\infty, \qquad (6)$$

we have

$$\begin{aligned}
&\limsup_{n\to\infty}\big(n\sum_{i=2}^{s+1}(1-F(u_n))^{i+s} + n\sum_{i=s+2}^{[n/k]}(1-F(u_n))^{2+2s}\big) = \\
=\ &\limsup_{n\to\infty}\Big[\Big(n^{\frac{1}{s+1}}(1-F(u_n))\Big)^{s+1}(1-F(u_n))\frac{1-(1-F(u_n))^s}{F(u_n)} + \qquad (7)\\
&+\Big(n^{\frac{1}{s+1}}(1-F(u_n))\Big)^{s+1}([n/k]-s-1)(1-F(u_n))^{1+s}\Big] = \\
=\ &\tfrac{\tau^{2+2s}}{k} \to 0 \text{ , as } k\to\infty.
\end{aligned}$$

The conclusion is immediate by (5), (6) and (7). Therefore, if we choose levels u_n such that (6) holds or, equivalently, such that $n(1-F_Y(u_n))\to\tau$, as $n\to\infty$ with F_Y the common d.f. of $Y_1, Y_2, ...$, then the maximum of $\{Y_i\}$ has a similar asymptotic behaviour as the one considered in a independent context and $\theta=1$.

Now we focus on the tail index computation. If we consider again u_n satisfying (6) and such that $u_n\equiv u_n(x)=a_nx+b_n$, with $\{a_n>0\}$ and $\{b_n\}$ real sequences, then, taking $\tau\equiv\tau(x)$, we have

$$n^{1/(s+1)}\big(1-F(a_nx+b_n)\big)\sim\tau(x) \text{ , } 0<\tau(x)<\infty. \qquad (8)$$

The i.i.d. assumption for $\{X_i\}$ combined with (8) leads to

$$n(1-F_Y(a_nx+b_n)) = \Big(n^{1/(s+1)}\big(1-F(a_nx+b_n)\big)\Big)^{s+1}\sim\tau(x)^{s+1}. \qquad (9)$$

Assume $M_n^Y=\max(Y_1,...,Y_n)$. As $F\in\mathcal{D}(G_\gamma)$, there are real sequences $\{a_n^*>0\}$ and $\{b_n^*\}$ such that $P\big(\hat{M}_n\le a_n^*x+b_n^*\big)\to G(x)$ as $n\to\infty$ or, equivalently,

$$n(1-F(a_n^*x+b_n^*)) \underset{n\to\infty}{\longrightarrow} -\ln G(x), \qquad (10)$$

where $G(x)$ is a *GEV*. We have that conditions $\mathrm{D}(a_nx+b_n)$ and $\mathrm{D}'(a_nx+b_n)$ both hold for $\{Y_i\}$. Applying Theorem 2.5 and relation (9) then, as $n\to\infty$,

$$P\big(M_n^Y \le a_n x + b_n\big) \to e^{-\tau(x)^{s+1}},$$

or,

$$P\big(M_n^Y \le a_n x + b_n\big) \to e^{-(-\ln G(x))^{s+1}}, \tag{11}$$

since (10) also holds replacing n by $n^{1/(s+1)}$ and considering $a^*_{[n^{1/(s+1)}]} = a_n$ and $b^*_{[n^{1/(s+1)}]} = b_n$. Now, observe that

$$e^{-(-\ln G(x))^{s+1}} = e^{-(1+\gamma x)^{-(s+1)/\gamma}} = e^{-(1+\frac{\gamma}{s+1}x(s+1))^{-1/(\gamma/(s+1))}}. \tag{12}$$

From (11) and (12) we conclude

$$P\Big(M_n^Y \le \widetilde{a}_n x + b_n\Big) \to exp\Big(-\big(1+\gamma^\star x\big)^{-1/\gamma^\star}\Big)\text{ , as } n \to \infty,$$

with $\gamma^\star = \gamma/(s+1)$ and $\widetilde{a}_n = a_n/(s+1)$. Hence $F_Y \in \mathcal{D}(G_{\gamma^\star})$.

3.2. $\{X_i\}$ stationary with a weak dependence structure

In this section we prove that the weak local dependence conditions D, D′ and D′′ hold for $\{Y_i\}$, whenever they hold for $\{X_i\}$, respectively. $\{Y_i\}$ is obviously a stationary sequence due to the definition of Y_i and the stationarity of $\{X_i\}$. We begin by stating the following lemma which is an important tool to prove the above assertion with respect to condition D.

Lemma 3.1. *Suppose that $D(u_n)$ holds for $\{X_i\}$ for a given real numbers sequence $\{u_n\}$, i.e., for any integer sets $I = \big\{i_1, ..., i_p\big\}$ and $J = \big\{j_1, ..., j_{p'}\big\}$ such that*

$$1 \le i_1 < ... < i_p < j_1 < ... < j_{p'} \le n \quad \textit{and} \quad j_1 - i_p \ge \ell, \tag{13}$$

we have $|F^X_{i_1,...,i_p,j_1,...,j_{p'}}(u_n) - F^X_{i_1,...,i_p}(u_n)F^X_{j_1,...,j_{p'}}(u_n)| \le \alpha_{n,\ell}$ with

$$\alpha_{n,\ell_n} \underset{n\to\infty}{\longrightarrow} 0, \tag{14}$$

for some sequence $\ell_n = o(n)$. Then

$$|\overline{F}^X_{i_1,...,i_p,j_1,...,j_{p'}}(u_n) - \overline{F}^X_{i_1,...,i_p}(u_n)\overline{F}^X_{j_1,...,j_{p'}}(u_n)| \le \alpha^\star_{n,\ell} \tag{15}$$

with $\alpha^\star_{n,\ell_n} \underset{n\to\infty}{\longrightarrow} 0$.

Proof. Writing the first member of (15) using the complementary and after some calculations, we have

$$
\begin{aligned}
&\left|\overline{F}^X_{i_1,\ldots,i_p,j_1,\ldots,j_{p'}}(u_n)-\overline{F}^X_{i_1,\ldots,i_p}(u_n)\overline{F}^X_{j_1,\ldots,j_{p'}}(u_n)\right|\leq\\
\leq\ &\sum_{i\in I,j\in J}\left|F^X_{i,j}(u_n)-F^X_i(u_n)F^X_j(u_n)\right|+\sum_{i\in I;\ j,j'\in J;\ j<j'}\left|F^X_{i,j,j'}(u_n)-F^X_i(u_n)F^X_{j,j'}(u_n)\right|+\\
&+\ldots+\sum_{i\in I}\left|F^X_{i,j_1,\ldots,j_{p'}}(u_n)-F^X_i(u_n)F^X_{j_1,\ldots,j_{p'}}(u_n)\right|+\\
&+\sum_{i,i'\in I;\ j\in J;\ i<i'}\left|F^X_{i,i',j}(u_n)-F^X_{i,i'}(u_n)F^X_j(u_n)\right|+\\
&+\ldots+\sum_{i,i'\in I;\ i<i'}\left|F^X_{i,i',j_1,\ldots,j_{p'}}(u_n)-F^X_{i,i'}(u_n)F^X_{j_1,\ldots,j_{p'}}(u_n)\right|+\ldots+\\
&+\sum_{j\in J}\left|F^X_{i_1,\ldots,i_p,j}(u_n)-F^X_{i_1,\ldots,i_p}(u_n)F^X_j(u_n)\right|+\\
&+\sum_{j,j'\in J;\ j<j'}\left|F^X_{i_1,\ldots,i_p,j,j'}(u_n)-F^X_{i_1,\ldots,i_p}(u_n)F^X_{j,j'}(u_n)\right|\\
&+\ldots+\left|F^X_{i_1,\ldots,i_p,j_1,\ldots,j_{p'}}(u_n)-F^X_{i_1,\ldots,i_p}(u_n)F^X_{j_1,\ldots,j_{p'}}(u_n)\right|
\end{aligned}
$$

Note that all parcels have r.v.'s that are distant (in time), at least, ℓ integers. So, using (13), we can bound each one of them by $\alpha_{n,\ell}$ with $\alpha_{n,\ell}$ verifying the conditions in (14). We have as much parcels as the number of different possible combinations of, at least, one element from I with, at least, one element from J. Denoting this number by $c_{p,p'}$ we have then

$$\left|\overline{F}^X_{i_1,\ldots,i_pj_1\ldots j_{p'}}(u_n)-\overline{F}^X_{i_1,\ldots,i_p}(u_n)\overline{F}^X_{j_1,\ldots,j_{p'}}(u_n)\right|\leq\alpha^\star_{n,\ell},\tag{16}$$

with $\alpha^\star_{n,\ell}=c_{p,p'}\alpha_{n,\ell}\to 0$, as $n\to\infty$. $\square$

Proposition 3.2. *Under the assertions of Lemma 3.1, condition $D(u_n)$ holds for $\{Y_i\}$.*

Proof. Considering the notation stated in Section 2, we have that

$$
\begin{aligned}
\overline{F}^Y_{i_1,\ldots,i_m}(u_n) &= P((X_{i_1},\ldots,X_{i_1+s},\ldots,X_{i_m},\ldots,X_{i_m+s})>u_n)\\
&= \overline{F}^X_{i_1,\ldots,i_1+s,\ldots,i_m,\ldots,i_m+s}(u_n)
\end{aligned}\tag{17}
$$

As D(u_n) holds for $\{X_i\}$ then (13) also holds for some $\ell^{(X)}=o(n)$. Consider any integer sets $I=\{i_1,\ldots i_p\}$ and $J=\{j_1,\ldots j_{p'}\}$ such that, for a given n, we have $1\leq i_1<\ldots<i_p<j_1<\ldots<j_{p'}\leq n$, and $j_1-(i_p+s)\geq\ell^{(X)}$ (i.e., $j_1-i_p\geq\ell^{(X)}+s=:\ell^{(Y)}$). An analogous reasoning used in the proof of Lemma 3.1 and an application of (17), lead us to

$$\left|F^{Y}_{i_1,...,i_p,j_1,...,j_{p'}}(u_n)-F^{Y}_{i_1,...,i_p}(u_n)F^{Y}_{j_1,...,j_{p'}}(u_n)\right|\leq$$
$$\leq \sum_{i\in I, j\in J}\left|\overline{F}^{X}_{i,...,i+s,j,...,j+s}(u_n)-\overline{F}^{X}_{i,...,i+s}(u_n)\overline{F}^{X}_{j,...,j+s}(u_n)\right|$$
$$+\sum_{i\in I;\ j<j'\in J}\left|\overline{F}^{X}_{i,...,i+s,j,...,j+s,j',...,j'+s}(u_n)-\overline{F}^{X}_{i,...,i+s}(u_n)\overline{F}^{X}_{j,...,j+s,j',...,j'+s}(u_n)\right|$$
$$+...+\sum_{i\in I}\left|\overline{F}^{X}_{i,...,i+s,j_1,...,j_{p'}+s}(u_n)-\overline{F}^{X}_{i,...,i+s}(u_n)\overline{F}^{X}_{j_1,...,j_{p'}+s}(u_n)\right|$$
$$+\sum_{i<i'\in I;\ j\in J}\left|\overline{F}^{X}_{i,...,i+s,i',...,i'+s,j,...,j+s}(u_n)-\overline{F}^{X}_{i,...,i+s,i',...,i'+s}(u_n)\overline{F}^{X}_{j,...,j+s}(u_n)\right|$$
$$+...+\sum_{i<i'\in I}\left|\overline{F}^{X}_{i,...,i+s,i',...,i'+s,j_1,...,j_{p'}+s}(u_n)-\overline{F}^{X}_{i,...,i+s,i',...,i'+s}(u_n)\overline{F}^{X}_{j_1,...,j_{p'}+s}(u_n)\right|$$
$$+...+\sum_{j\in J}\left|\overline{F}^{X}_{i_1,...,i_p+s,j,...,j+s}(u_n)-\overline{F}^{X}_{i_1,...,i_p+s}(u_n)\overline{F}^{X}_{j,...,j+s}(u_n)\right|+$$
$$+\sum_{j<j'\in J}\left|\overline{F}^{X}_{i_1,...,i_p+s,j,...,j+s,j',...,j'+s}(u_n)-\overline{F}^{X}_{i_1,...,i_p+s}(u_n)\overline{F}^{X}_{j,...,j+s,j',...,j'+s}(u_n)\right|$$
$$+...+\left|\overline{F}^{X}_{i_1,...,i_p+s,j_1,...,j_1+s,...,j_{p'}+s}(u_n)-\overline{F}^{X}_{i_1,...,i_p+s}(u_n)\overline{F}^{X}_{j_1,...,j_{p'}+s}(u_n)\right|.$$

Note that, in each parcel, r.v.'s distant, at least, $\ell^{(X)}$ integers from each other and so, we can bound them by $\alpha^{\star}_{n,\ell^{(X)}}$. Note also that the number of parcels is exactly the same obtained in Lemma 3.1 proof, i.e., $c_{p,p'}$. Therefore,

$$\left|F^{Y}_{i_1,...,i_p,j_1,...,j_{p'}}(u_n)-F^{Y}_{i_1,...,i_p}(u_n)F^{Y}_{j_1,...,j_{p'}}(u_n)\right|\leq \alpha^{\star\star}_{n,\ell^{(Y)}},$$

where, by (16), we have then

$$\alpha^{\star\star}_{n,\ell^{(Y)}}=c_{p,p'}\alpha^{\star}_{n,\ell^{(X)}}=c^2_{p,p'}\alpha_{n,\ell^{(X)}}\underset{n\to\infty}{\longrightarrow}0, \tag{18}$$

with $\ell^{(Y)}=\ell^{(X)}+s=o(n)$. □

Proposition 3.3. *If condition $D'(u_n)$ holds for $\{X_i\}$, then it also holds for $\{Y_i\}$.*

Proof. Assuming that condition $\mathrm{D}'(u_n)$ holds for $\{X_i\}$, we have that

$$\begin{aligned}
&\limsup_{n\to\infty} n\sum_{j=2}^{[n/k]} P\big(Y_1>u_n, Y_j>u_n\big)=\\
&=\limsup_{n\to\infty} n\sum_{j=2}^{[n/k]} P\big((X_1,...,X_{1+s},X_j,...,X_{j+s})>u_n\big)\\
&=\limsup_{n\to\infty} n\sum_{j=2}^{[n/k]} \big[P\big((X_2,...,X_{1+s},X_{j+1},...,X_{j+s})>u_n|\\
&\qquad |(X_1,X_j)>u_n\big)P\big((X_1,X_j)>u_n\big)\big]\\
&\leq \limsup_{n\to\infty} n\sum_{j=2}^{[n/k]} P\big(X_1>u_n, X_j>u_n\big)=0,
\end{aligned}$$

as $k\to\infty$ and so, it also holds for $\{Y_i\}$. □

Proposition 3.4. *Suppose that conditions of Lemma 3.1 and condition $D''(u_n)$ hold for $\{X_i\}$, for some integers k_n satisfying (4). Then condition $D''(u_n)$ also holds for $\{Y_i\}$.*

Proof. By Proposition 3.2 we conclude that $\{Y_i\}$ verifies condition $\mathrm{D}(u_n)$ too and, by (18) and (4) it is immediate that

$$k_n \to \infty \ , \quad k_n \alpha^{\star\star}_{n,\ell_n^{(Y)}} \to 0 \ , \quad k_n l_n^{(Y)}/n \to 0.$$

Considering again (4), then

$$\begin{aligned} k_n(1 - F_{Y_i}(u_n)) &= k_n P\big((X_i, ..., X_{i+s}) > u_n\big) \\ &= k_n P\big((X_{i+1}, ..., X_{i+s}) > u_n | X_i > u_n\big) P\big(X_i > u_n\big) \\ &\leq k_n P\big(X_i > u_n\big) \to_{n\to\infty} 0. \end{aligned}$$

An analogous reasoning used in Proposition 3.3 lead us to

$$\lim_{n\to\infty} \ n \sum_{j=2}^{[n/k]-1} \ P\big(Y_1 > u_n, Y_j \leq u_n < Y_{j+1}\big) = 0$$

Hence $\mathrm{D}''(u_n)$ holds for $\{Y_i\}$. □

Remark 3.1 The tail behaviour of $\{Y_i\}$ depends on the distribution of X_i as well as on its dependence structure.

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