How to compute the extremal index of stationary random fields

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Abstract: We present local dependence conditions for stationary random fields under which the extremal index and the asymptotic distribution of the maximum $M_{(n_1,\ldots,n_d)}$ can be calculated from the joint distribution of a finite number $s_1 s_2$ of variables.
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1 Introduction

Let $\mathbf{X} = \{X_{\mathbf{n}}\}_{\mathbf{n} \geq \mathbf{1}}$ be a stationary random field on $\mathbb{Z}_+^d$, where $\mathbb{Z}_+$ is the set of all positive integers and $d \geq 2$. We shall consider the conditions and results for $d = 2$ since it is notationally simplest and the proofs for higher dimensions follow analogous arguments.

For a family of real levels $\{u_{\mathbf{n}}\}_{\mathbf{n} \geq \mathbf{1}}$ and a subset $\mathbf{I}$ of the rectangle of points $\mathbf{R_n} = \{1, \ldots, n_1\} \times \{1, \ldots, n_2\}$, we will denote the event $\{X_{\mathbf{i}} \leq u_{\mathbf{n}} : \mathbf{i} \in \mathbf{I}\}$ by $M_{\mathbf{n}}(\mathbf{I})$ or simply by $M_{\mathbf{n}}$ when $\mathbf{I} = \mathbf{R_n}$. If $\mathbf{I} = \emptyset$ then $M_{\mathbf{n}}(\mathbf{I}) = -\infty$.

For each $i = 1, 2$, we say the pair $\mathbf{I}$ and $\mathbf{J}$ is in $\mathcal{S}_i(l)$ if the distance between $\Pi_i(\mathbf{I})$ and $\Pi_i(\mathbf{J})$ is great or equal to l, where Π_i, $i = 1, 2$ denote the cartesian projections. The distance $d(\mathbf{I}, \mathbf{J})$ between sets $\mathbf{I}$ and $\mathbf{J}$ of $\mathbb{Z}_+^d$, $d \geq 1$, is the minimum of distances $d(\mathbf{i}, \mathbf{j}) = max\{|i_s - j_s|, s = 1, \ldots, d\}$, $\mathbf{i} \in \mathbf{I}$ and $\mathbf{j} \in \mathbf{J}$.

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Suppose that $\mathbf{X}$ satisfies the coordinatewise-mixing $\Delta(u_{\mathbf{n}})$-condition introduced in Leadbetter and Rootzén (1998) which exploit the past and future separation one coordinate at a time. Therefore, we suppose that for $\mathbf{X}$ there exist sequences of integer valued constants $\{k_{n_i}\}$, $\{l_{n_i}\}$, $i = 1, 2$ such that, as $\mathbf{n} \to \infty$, we have $(k_{n_1}, k_{n_2}) \to \infty$, $(\frac{k_{n_1} l_{n_1}}{n_1}, \frac{k_{n_2} l_{n_2}}{n_2}) \to \mathbf{0}$ and $(k_{n_1}\Delta_1, k_{n_1} k_{n_2} \Delta_2) \to \mathbf{0}$, where Δ_i are the components of the mixing coefficient defined as follows:

$$\Delta_1 = sup|P(M_{\mathbf{n}}(\mathbf{I}_1) \le u_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{I}_2) \le u_{\mathbf{n}}) - P(M_{\mathbf{n}}(\mathbf{I}_1) \le u_{\mathbf{n}}) P(M_{\mathbf{n}}(\mathbf{I}_2) \le u_{\mathbf{n}})|,$$

where the supremum is taken over pairs of $\mathbf{I}_1$ and $\mathbf{I}_2$ in $\mathcal{S}_1(l_{n_1})$ such that $|\Pi_1(\mathbf{I}_2)| \le \frac{n_1}{k_{n_1}}$,

$$\Delta_2 = sup|P(M_{\mathbf{n}}(\mathbf{I}_1) \le u_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{I}_2) \le u_{\mathbf{n}}) - P(M_{\mathbf{n}}(\mathbf{I}_1) \le u_{\mathbf{n}}) P(M_{\mathbf{n}}(\mathbf{I}_2) \le u_{\mathbf{n}})|,$$

where the supremum is taken over pairs of $\mathbf{I}_1$ and $\mathbf{I}_2$ in $\mathcal{S}_2(l_{n_2})$ such that $\Pi_1(\mathbf{I}_1) = \Pi_1(\mathbf{I}_2)$ and $|\Pi_2(\mathbf{I}_2)| \le \frac{n_2}{k_{n_2}}$.

Under the coordinatewise-mixing $\Delta(u_{\mathbf{n}})$-condition we have the following asymptotic independence for maxima over $\{(i-1)[\frac{n_1}{k_{n_1}}]+1, \dots, i[\frac{n_1}{k_{n_1}}]\} \times \{(j-1)[\frac{n_2}{k_{n_2}}]+1, \dots, j[\frac{n_2}{k_{n_2}}]\}$, $i = 1, \dots, k_{n_1}$, $j = 1, \dots, k_{n_2}$:

$$P(M_{\mathbf{n}} \le u_{\mathbf{n}}) - P^{k_{n_1} k_{n_2}}(M_{([\frac{n_1}{k_{n_1}}],[\frac{n_2}{k_{n_2}}])} \le u_{\mathbf{n}}) \longrightarrow_{\mathbf{n} \to \infty} 0. \tag{1.1}$$

Accordingly Choi (2002), we shall say that $\mathbf{X}$ has extremal index θ if for each $\tau > 0$ there exists $\{u_{\mathbf{n}}^{(\tau)}\}_{\mathbf{n} \ge \mathbf{1}}$ such that, as $\mathbf{n} \to \infty$, $n_1 n_2 P(X_{\mathbf{1}} > u_{\mathbf{n}}^{(\tau)}) \to \tau$ and $P(M_{\mathbf{n}} \le u_{\mathbf{n}}^{(\tau)}) \to e^{-\tau}$.

This paper is concerned with describing how to compute the extremal index of the stationary random field under local mixing conditions analogous to those considered in Chernick et al. (1991).

Those local restrictions on the oscillations of the values of the random field enable to compute the extremal index from the joint distribution of a finite number $s_1 s_2$ of variables.

2 Condition $D^{(\mathbf{s})}(u_{\mathbf{n}})$

In the following consider $\mathbf{R}^*_{\mathbf{i},\mathbf{j}} = \{i_1, i_1+1, \ldots, j_1\} \times \{i_2, i_2+1, \ldots, j_2\} - \{\mathbf{i}\}$. In particular, for $\mathbf{i} = \mathbf{1}$ we write simply $\mathbf{R}^*_{\mathbf{j}}$. For sake of simplicity we write $[\mathbf{n}/\mathbf{k}]$ for $([\frac{n_1}{k_{n_1}}], [\frac{n_2}{k_{n_2}}])$.

Definition The random field $\mathbf{X}$ satisfies the condition $D^{(\mathbf{s})}(u_{\mathbf{n}})$, for some $s \in \mathbb{Z}^2_+$, if there exist sequences of integer valued constants $\{k_{n_i}\}$, $\{l_{n_i}\}$, $i = 1, 2$, such that, as $\mathbf{n} \to \infty$, we have $(k_{n_1}, k_{n_2}) \to \infty$, $(\frac{k_{n_1} l_{n_1}}{n_1}, \frac{k_{n_2} l_{n_2}}{n_2}) \to \mathbf{0}$ and

$$n_1 n_2 \sum_{\substack{\mathbf{j} \leq [\mathbf{n}/\mathbf{k}] \\ d(\mathbf{1},\mathbf{j}) \geq max\{s_1, s_2\}}} P(X_{\mathbf{1}} > u_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{R}^*_{\mathbf{s}}) \leq u_{\mathbf{n}}, X_{\mathbf{j}} > u_{\mathbf{n}}) \to 0.$$

For the cases $\mathbf{s} = \mathbf{1} = (1,1)$ and $\mathbf{s} = \mathbf{2} = (2,2)$ we find in the above definition the local conditions considered in Pereira and Ferreira (2005). We shall pursue the direction of this dependence conditions and extend to spatial processes some known formulas to obtain the extremal index of time series .

Proposition 2.1 *Let $\mathbf{X}$ be a stationary random field satisfying the $\Delta(u_{\mathbf{n}})$-condition and $D^{(\mathbf{s})}(u_{\mathbf{n}})$-condition for some $\mathbf{s}$. As $\mathbf{n} \to \infty$, we have*

$$n_1 n_2 P(X_{\mathbf{1}} > u_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{R}^*_{\mathbf{s}}) \leq u_{\mathbf{n}}) \to \nu > 0$$

if only if

$$P(M_{\mathbf{n}} \leq u_{\mathbf{n}}) \to e^{-\nu}, \ \nu > 0.$$

Proof: From (1.1) we get

$$P(M_{\mathbf{n}} \leq u_{\mathbf{n}}) - \Big(P(M_{[\mathbf{n}/\mathbf{k}]} \leq u_{\mathbf{n}})\Big)^{k_{n_1} k_{n_2}} =$$

$$P(M_{\mathbf{n}} \leq u_{\mathbf{n}}) - exp\Big(-(1+o(1)) k_{n_1} k_{n_2} P(M_{[\mathbf{n}/\mathbf{k}]} > u_{\mathbf{n}})\Big) =$$

$$P(M_{\mathbf{n}} \leq u_{\mathbf{n}}) - exp\Big(-(1+o(1)) k_{n_1} k_{n_2} \sum_{\mathbf{i} \leq [\mathbf{n}/\mathbf{k}]} P(X_{\mathbf{i}} > u_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{R}^*_{\mathbf{i},[\mathbf{n}/\mathbf{k}]}) \leq u_{\mathbf{n}})\Big) =$$

$$P(M_{\mathbf{n}} \leq u_{\mathbf{n}}) - exp\Big(-(1+o(1))k_{n_1}k_{n_2} \sum_{\mathbf{i}\leq[\mathbf{n}/\mathbf{k}]} P(X_{\mathbf{1}} > u_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{R}^*_{[\mathbf{n}/\mathbf{k}]-\mathbf{i}+\mathbf{1}}) \leq u_{\mathbf{n}})\Big) = o(1).$$

The result follows now by applying stationarity and the $D^{(\mathbf{s})}(u_{\mathbf{n}})$-condition, since for $[\frac{n_1}{k_{n_1}}] - i_1 + 1 \geq s_1$ or $[\frac{n_2}{k_{n_2}}] - i_2 + 1 \geq s_2$ it holds

$$P(X_{\mathbf{1}} > u_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{R}^*_{\mathbf{s}}) \leq u_{\mathbf{n}}) - P(X_{\mathbf{1}} > u_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{R}^*_{[\mathbf{n}/\mathbf{k}]-\mathbf{i}+\mathbf{1}}) \leq u_{\mathbf{n}}) \leq$$

$$\sum_{\substack{\mathbf{j}\leq[\mathbf{n}/\mathbf{k}] \\ d(\mathbf{1},\mathbf{j})\geq max\{s_1,s_2\}}} P(X_{\mathbf{1}} > u_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{R}^*_{\mathbf{s}}) \leq u_{\mathbf{n}}, X_{\mathbf{j}} > u_{\mathbf{n}}) = o(n_1 n_2).$$

Therefore

$$P(M_{\mathbf{n}} \leq u_{\mathbf{n}}) - exp\Big(-(1+o(1))n_1 n_2 P(X_{\mathbf{1}} > u_{\mathbf{n}}, M_{\mathbf{n}}(\mathbf{R}^*_{s}) \leq u_{\mathbf{n}}) + o(1)\Big) = o(1).$$

□

As a corollary we provide a relation between the limit of $n_1 n_2 P(X_{\mathbf{1}} > u_{\mathbf{n}}^{(\tau)}, M_{\mathbf{n}}(\mathbf{R}^*_{\mathbf{s}}) \leq u_{\mathbf{n}}^{(\tau)})$ and the extremal index.

Proposition 2.2 *Let* $\mathbf{X}$ *be a stationary random field satisfying the* $\Delta(u_{\mathbf{n}}^{(\tau)})$*-condition and* $D^{(\mathbf{s})}(u_{\mathbf{n}}^{(\tau)})$*-condition for each tau* > 0*. The extremal index of* $\mathbf{X}$ *exists and is* θ *if and only if, as* $\mathbf{n} \to \infty$*, it holds*

$$n_1 n_2 P(M_{\mathbf{n}}(\mathbf{R}^*_{\mathbf{s}}) \leq u_{\mathbf{n}} | X_{\mathbf{1}} > u_{\mathbf{n}}) \to \theta.$$

From the Normal comparison lemma we can prove that the stationary normal random fields such that $\rho_{\mathbf{n}} = E(X_{\mathbf{j}} X_{\mathbf{j}+\mathbf{n}})$ satisfies $\rho_{\mathbf{n}} log(n_1 n_2) \to 0$, as $\mathbf{n} \to \infty$, satisfy $\Delta(u_{\mathbf{n}}^{(\tau)})$-condition and $D^{(\mathbf{1})}(u_{\mathbf{n}}^{(\tau)})$-condition (Pereira e Ferreira (2005)).

The m-dependent random fields satisfy $\Delta(u_{\mathbf{n}}^{(\tau)})$-condition and $D^{(\mathbf{s})}(u_{\mathbf{n}}^{(\tau)})$-condition for some $\mathbf{s}$.

It is an open and interesting question know if some autoregressive random fields can satisfy $D^{(\mathbf{s})}(u_{\mathbf{n}}^{(\tau)})$-condition for some $\mathbf{s}$ with diferent coordinates.

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